Listing triangles in expected linear time on a class of power law graphs

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Abstract

Enumerating triangles (3-cycles) in graphs is a kernel operation for social network analysis. For example, many community detection methods depend upon finding common neighbors of two related entities. We consider Cohen's simple and elegant solution for listing triangles: give each node a "bucket." Place each edge into the bucket of its endpoint of lowest degree, breaking ties consistently. Each node then checks each pair of edges in its bucket, testing for the adjacency that would complete that triangle. Cohen presents an informal argument that his algorithm should run well on real graphs. We formalize this argument by providing an analysis for the expected running time on a class of random graphs, including power law graphs.

We consider a rigorously defined method for generating a random simple graph, the erased configuration model (ECM). In the ECM each node draws a degree independently from a marginal degree distribution, endpoints pair randomly, and we erase self loops and multiedges. If the marginal degree distribution has a finite second moment, it follows immediately that Cohen's algorithm runs in expected linear time. Furthermore, it can still run in expected linear time even when the degree distribution has such a heavy tail that the second moment is not finite. We prove that Cohen's algorithm runs in expected linear time when the marginal degree distribution has finite $\frac{4}{3}$ moment and no vertex has degree larger than $\sqrt{n}$. In fact we give the precise asymptotic value of the expected number of edge pairs per bucket. A finite $\frac{4}{3}$ moment is required; if it is unbounded, then so is the number of pairs. The marginal degree distribution of a power law graph has bounded $\frac{4}{3}$ moment when its exponent α is more than $\frac{7}{3}$. Thus for this class of power law graphs, with degree at most $\sqrt{n}$, Cohen's algorithm runs in expected linear time. This is precisely the value of α for which the clustering coefficient tends to zero asymptotically, and it is in the range that is relevant for the degree distribution of the World-Wide Web.

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1 Introduction

Identifying triangles (3-cycles) in graphs is a kernel operation for social network analysis. *Listing* triangles can be more involved that *finding* one triangle or *counting* triangles [17–19]. Latapy [10] surveys the state of the art in all three variants. Tsourakakis [17] more specifically surveys results in counting or estimating the number of triangles. In this paper, we focus on listing triangles.

Although counting triangles is a fundamental problem in social network analysis [6, 7], we have found relatively few direct applications in the literature. Fudos and Hoffman [9] describe a graph-based approach for solving systems of geometric constraints. Their algorithm requires finding and handling each triangle in a graph. They claim applications of this geometric contraint processing in computer-aided design, stereochemistry, kinematic analysis of robots and other mechanisms, and robot motion planning. These graphs are sparse. However, we have little reason to believe they would be well represented by a class of random graphs whose degree distribution has a finite 43 moment. We are primarily interested in enumerating triangles because it is a fundamental step in detecting an edge support measure in a community detection algorithm we are actively using for social network analysis at Sandia National Laboratories [1].

Latapy gives several algorithms for listing triangles that minimize the constant in their $\Theta(m)$ space complexity, where m is the number of edges in the graph [10]. We are satisfied with that asymptotic space bound and allow ourselves $O(m)$ storage in addition to the graph representation.

We analyze a simple, elegant algorithm by Cohen [3] for listing triangles in graphs: Give each node a "bucket." Place each edge into the bucket of its endpoint of lowest degree, breaking ties consistently. Two edges (u, v) and (v, w) that share a vertex v might be part of a triangle (u, v, w). One need only check for the edge (u, w). Each node checks each pair of edges in its bucket, testing for the adjacency that would complete that triangle. Since each triangle has one vertex of lowest degree (with tie-breaking), precisely one of its vertices will find it.

For a graph $G = (V, E)$ with n nodes and m edges, placing nodes into buckets requires $O(m)$ time. The complexity of Cohen's algorithm is determined by the number of edge pairs, summed over all node buckets. Cohen informally argues that nodes of high degree donate many edges to nodes of low degree, and therefore the total work should be reasonable.

We analyze Cohen's algorithm for a class of random graphs with arbitrary degree sequences. We show that Cohen's algorithm runs in expected linear time for a class of power law graphs defined below, which also shows the expected number of triangles for this class of graphs is also linear.

1.1 Preliminaries

One way to define a random graph is by its underlying degree distribution. We can express this via a *marginal reference distribution*, which statistically gives the probability distribution for the degree of an individual node within the graph. We refer to such a distribution as a reference distribution. More specifically, for any node, its random degree D has a probability mass function

$$f(d) = P(D = d), \quad \text{integer } d \in [0, \infty), \tag{1}$$

where $\sum_{d=0}^{\infty} f(d) = 1$.

Currently *power law* degree distributions are popular, since they appear in many large data sets such as social networks, the AS-level internet graph [15], and the world-wide-web graph [11, 12]. The power-law degree distribution is defined on $[1, \infty)$ and with reference distribution

$$P(D = d) = f(d) = G(d)d^{-\alpha}, \quad \text{integer } d \geq 1, \tag{2}$$

where $\alpha \geq 1$ is the tail index and $G(\cdot)$ is a slowly varying function at ∞ (i.e., for any $t > 0$, $G(td)/G(d) \to 1$ as $d \to \infty$.) Our results apply for n-node power law graphs with $\alpha > \frac{7}{3}$ and maximum degree $\sqrt{n}$. In Liu et. al.'s model of WWW graph growth, when the out-degree has a power law distribution and average degree 3, then the indegree distribution follows a power law with $\alpha = \frac{7}{3}$.

The rth moment, $r > 0$, of the reference degree distribution is given by $\mathrm{E}[D^r] = \sum_{d=0}^{\infty} d^r \cdot f(d)$. The first moment ($r = 1$) is the mean, or average, degree.

Power-law graphs have bounded (constant) average degree (first moments) for values of α common in the literature (e.g. $2 < \alpha \leq 3$). Although one might be tempted to argue that then the expected value of the square of the average degree, and hence the expected number of pairs per bucket, is also bounded, this would not be correct. The rth moment calculation, in this case for $r = 2$ is nonlinear. Thus it's possible for larger moments to have limits that grow as a function of n because the weight of larger values of d becomes significant for larger r. The probability $f(d)$ is not sufficiently small relative to d^r. We will show in Section 2 that for distributions with finite second moment, Cohen's algorithm trivially runs in linear time.

1.2 Analysis of Power-Law Graphs

Latapy describes two triangle listing algorithms that run in worst-case $O(mn^{\frac{1}{\alpha}})$ time and $O(m)$ space, where n is the number of vertices in an undirected power-law graph, m is the number of edges, and the probability of a vertex having degree k is proportional to $k^{-\alpha}$ [10]. Although Latapy describes power-law degree distribution by probabilities, he defines an instance of a power law graph based upon the vector of degrees (*degree sequence*) for its n vertices. Specifically, he defines a *continuous* power law graph as one where the proportion of vertices of each degree is equal to the expected value. This creates correlation among the vertex degrees. It also limits the number of graphs considered to follow a power law. This is somewhat akin to considering a coin fair only if n tosses results in precisely $n/2$ heads and $n/2$ tails. Once defined this way, Latapy can consider worst-case placement of edges for his algorithm.

To do a statistically rigorous expected average-case analysis of Cohen's algorithm, we must consider, not just averages, but full joint-degree distributions. This is a probability distribution on the degree sequence for the entire graph, not a single specific degree sequence. In Section 1.4 we describe a rigorous graph generation process. We then analyze the expected performance of Cohen's algorithm over the distribution of graphs that process can generate. This allows us to use a node-centric analysis, taking advantage of the symmetry and independence of all nodes.

1.3 Results

Our primary contributions are:

1. We introduce a rigorous technique for analyzing the neighborhoods of vertices in random graphs using a realistic generation model. We require only that the marginal degree distribution has finite $\frac{4}{3}$ moment.
2. We apply this technique to n-node power law graphs with exponent $\alpha > \frac{7}{3}$ and degree at most $\sqrt{n}$, to prove linear expected time bounds on Cohen's triangle listing algorithm.
3. We provide experimental evidence of a small difference between the performance of Cohen's algorithm on power-law inputs and its predicted performance from the generation model.

1.4 Graph Generation Model

We generate simple, undirected random graphs with n nodes according to the "erased configuration model" (ECM) of Britton, Deijfen and Martin-Löf [16]. The ECM accepts an arbitrary degree distribution, for example a power-law degree distribution, as input. The ECM is a modification of the configuration model (CM) introduced by Bender and Canfield [4] and studied by Wormald [21], Bollobás [2] and Newman [14] among others. In particular, we employ a version of the ECM which involves truncation of an underlying degree distribution to ensure degrees are essentially bounded by n^τ for some $\tau \in (0, 1/2)$ (cf. Britton, Deijfen and Martin-Löf [16], p. 1381). To ensure a physically meaningful or well behaved degree sequence, many authors in the physics community have used similar truncation. Malloy and Reed [13] truncate at both $n^{1/4-\epsilon}$ and $n^{1/8-\epsilon}$, Newman [14] truncates at the maximum degree, and Chung and Lu [5] truncate such that $\max_i d_i^2 \leq \sum_i d_i$. Although we consider ECM graphs with degree truncation, we shall simply refer to the final random graph as an ECM graph.

Suppose we are given an underlying degree distribution (1). We require the degree distribution to have a finite first moment or expected value $\mathrm{E}[D] = \sum_{d=0}^{\infty} d \cdot f(d) = O(1)$. To generate an ECM graph with n nodes, we draw potential degrees for each node from a truncated version of (1):

$$f_n(d) \equiv \begin{cases} f(d)/C_n & \text{integer } 0 \leq d \leq L(n)n^\tau \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $C_n \equiv \sum_{0 \leq d \leq L(n)n^\tau} f(d)$ is the normalizing constant, the power $\tau \in (0, 1/2)$ may be arbitrarily chosen, and $L(\cdot)$ represents an arbitrary function that is slowly varying at ∞ (i.e., $\lim_{n\to\infty} L(nt)/L(n) = 1$ for any $t > 0$ such as $L(n) = O(log^k(n))$ for constant k. Given n, let random variables $D_{1,n}, \ldots, D_{n,n}$ be independent, identically distributed draws from (3) and assign $D_{i,n}$ "stubs" to node $i = 1, \ldots, n$. Assuming $S_n \equiv \sum_{i=1}^{n} D_{i,n}$ is even, randomly pair the S_n stubs, with every "pairing configuration" equally likely. If S_n is odd, before randomly pairing, randomly pick one integer I from $\{1, \ldots, n\}$, and increment $D_{I,n}$ by 1. This process generates a graph under the CM, which may not be simple due to self-loops (formed from randomly pairing stubs on the same node) or multi-edges (formed by pairing multiple stubs between two nodes). To obtain a simple graph, remove all self-loops and merge any existing multi-edges into a single edge to produce a final graph under the ECM.

After the erasure step, denote the observed degrees in the final simple graph as $D^s_{1,n}, \ldots, D^s_{n,n}$, which have a common distribution $P(D^s_{1,n} = d)$, $d \geq 0$, that depends on the number of nodes n in the graph as well as the reference degree distribution f from (1). When f has a finite mean, Theorem 2.1 of Britton, Deijfen and Martin-Löf (2006) implies that, for any integer d,

$$\lim_{n\to\infty} P(D^s_{1,n} = d) = P(D = d) = f(d),$$

implying that the degree distribution of an ECM graph will asymptotically match the reference distribution (1). We note that Britton, Deijfen and Martin-Löf [16] describe other random graphs based on the CM, such as a repeated CM (i.e., repeatedly generating CM graphs until a simple one is obtained). But, for the degree distribution in a repeated CM graph to asymptotically match a reference degree distribution f, we would need to assume f has a finite second moment which is unnecessary and restrictive in the results to follow.

2 Bounding the Work in Cohen's Algorithm

We now establish a non-trivial bound on the expected sizes of "buckets" in Cohen's triangle enumeration algorithm for large ECM graphs.

We define nodes i and j be neighbors if they share an edge in the ECM graph. Fix an arbitrary node i among the n nodes of the graph and define a "bucket"

$$\mathcal{B}_{i,n} = \{j : i \neq j, D^s_{i,n} \leq D^s_{j,n}, \text{node } i \text{ and node } j \text{ are neighbors}\},$$

which corresponds to the set of all neighbors of node i having degree at least as great as that of node i. This produces buckets somewhat larger than Cohen's algorithm, since here an edge between nodes of equal degree count toward the bucket of both endpoints, while in Cohen's algorithm, it will be in only one. Let $N_{i,n} = |\mathcal{B}_{i,n}|$ be the size of the bucket for node i in a graph with n nodes. The number of possible node pairs that can be formed from nodes in the bucket $\mathcal{B}_{i,n}$ is

$$\binom{N_{i,n}}{2} = \frac{N_{i,n}(N_{i,n}-1)}{2}.$$

We wish to bound the expected value $\mathrm{E}[\binom{N_{i,n}}{2}]$ as the number of nodes $n \to \infty$. Recall that the rth moment, $r > 0$, of the reference degree distribution is given by $\mathrm{E}[D^r] = \sum_{d=0}^{\infty} d^r \cdot f(d)$ and first moment ($r = 1$) is the mean degree. Our main theorem gives an explicit expression for $\lim_{n\to\infty} \mathrm{E}[\binom{N_{i,n}}{2}]$ when the underlying degree distribution (1) has a finite 4/3-moment. We prove this upper bound on the expected triangle-searching work per bucket in the Cohen algorithm is finite, bounded by a constant. We also determine the limiting form of expected bucket size $\mathrm{E}[N_{i,n}]$. For the Cohen algorithm, the expected bucket size is one half the expected number of edges in the ECM. For our slightly more pessimistic counting, it will be no more than twice this value. The proof of Theorem 1 appears in Section 3.

Theorem 1. *Under the ECM for generating random graphs (involving truncation of the reference degree distribution (1)), suppose* $\mathrm{E}[D] = \sum_{d=0}^{\infty} d \cdot f(d) \in (0, \infty)$.
(i) Then as $n \to \infty$,

$$\mathrm{E}[N_{i,n}] \to \frac{1}{\mathrm{E}[D]} \sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} d_1 d_2 f(d_1) f(d_2) = O(1).$$

(ii) If $\mathrm{E}[D^{4/3}] = O(1)$ *for (1), then as* $n \to \infty$,

$$\mathrm{E}\left[\binom{N_{i,n}}{2}\right] \to \frac{1}{2(\mathrm{E}[D])^2} \sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} d_1(d_1-1) d_2 d_3 f(d_1) f(d_2) f(d_3) = O(1).$$

Remark 1: We wish to highlight one crucial point regarding moments, degree truncation and Theorem 1. In subsequent work outside the scope of this paper [8], we show that if the reference degree distribution f in (1) has a finite second moment $\mathrm{E}[D^2] = \sum_{d=0}^{\infty} d^2 \cdot f(d) < \infty$, both limit results in Theorem 1 hold *without any degree truncation* in the ECM. That is, in this case, the ECM graph could be formed by initial stubs $D_{1,n}, \ldots, D_{n,n}$ drawn directly from f in (1), as opposed to the truncated version f_n in (3), and the same limits for the bucket expectations remain valid. Consequently, the limit results in Theorem 1 apply to all ECM graphs where $\mathrm{E}[D^2] = O(1)$ even without truncation. The moment assumption $\mathrm{E}[D^{4/3}] = O(1)$ turns out to be bare minimal for ensuring finite limiting bucket expectations in Theorem 1 (just enough to guarantee $\sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} d_1(d_1-1) d_2 d_3 f(d_1) f(d_2) f(d_3)$ is finite for example), but as a trade-off we require additional degree truncation (3) to establish such limits under this weak moment condition.

To frame the results in Theorem 1, note that the bucket size $N_{i,n}$ of node i is always bounded by the initial degree $D_{i,n}$ of node i; that is, $N_{i,n} \leq D_{i,n}$ and $\binom{N_{i,n}}{2} \leq \binom{D_{i,n}}{2} \leq D_{i,n}^2$. Consequently, it follows that

$$\limsup_{n\to\infty} \mathrm{E}\left[\binom{N_{i,n}}{2}\right] \leq \limsup_{n\to\infty} \mathrm{E}[D_{i,n}^2] = \limsup_{n\to\infty} \sum_{d=0}^{\infty} d^2 f_n(d) = \sum_{d=0}^{\infty} d^2 f(d) \equiv \mathrm{E}[D^2].$$

Hence, whenever $\mathrm{E}[D^2] = O(1)$ in the reference degree distribution (1), the value of $\mathrm{E}[\binom{N_{i,n}}{2}]$ will be trivially finite and bounded for all graph sizes n (though the exact form of $\lim_{n\to\infty} \mathrm{E}[\binom{N_{i,n}}{2}]$ does not easily follow). A crucial nontrivial aspect of Theorem 1(ii) is that the limit of $\mathrm{E}[\binom{N_{i,n}}{2}]$ can also be finite even in cases where the reference degree distribution f is so heavy in its tail probabilities that $\mathrm{E}[D^2] = \omega(1)$. In this situation, $\mathrm{E}[D^2]$ no longer provides a trivial finite bound on the limiting value of $\mathrm{E}[\binom{N_{i,n}}{2}]$. These conditions hold for power-law degree distributions (2) with exponent (or tail index) in particular ranges. We can re-cast the results of Theorem 1 for the special case of power laws (2) under the additional assumption that the slowly varying function is bounded away from zero (e.g., $G(d) = C$ or $G(d) = C \log(d + 1)$ for some $C > 0$).

Corollary 1. *Under the erased configuration model, suppose the degree distribution is a power law* (2) *where the slowly varying function satisfies* $\liminf_{d\to\infty} G(d) > 0$. *Then, the following table summaries moments and limits as finite (F) or infinite* (∞). *"F-Th1" means the values of finite limits are given in Theorem 1.*

	$\mathrm{E}[D]$	$\lim_{n\to\infty} \mathrm{E}[N_{i,n}]$	$\mathrm{E}[D^{4/3}]$	$\lim_{n\to\infty} \mathrm{E}\left[\binom{N_{i,n}}{2}\right]$	$\mathrm{E}[D^2]$
$\alpha \le 2$	∞	∞	∞	∞	∞
$\alpha \in (2, 2\frac{1}{3}]$	*F*	*F-Th1*	∞	∞	∞
$\alpha \in (2\frac{1}{3}, 3]$	*F*	*F-Th1*	*F*	*F-Th1*	∞
$\alpha > 3$	*F*	*F-Th1*	*F*	*F-Th1*	*F*

Corollary 1 implies that, for power laws (2), the limiting expectation in Theorem 1(ii) [or (i)] is finite if and only if $\mathrm{E}[D^{4/3}] = O(1)$ [$\mathrm{E}[D] = O(1)$], which holds if and only if $\alpha > 2\frac{1}{3}$ [$\alpha > 2$]. Perhaps surprisingly, the expected pairs from a bucket will remain finite as the graph grows for heavy tailed power laws with index $\alpha \in (2\frac{1}{3}, 3]$ for which $\mathrm{E}[D^2] = \omega(1)$ holds.

The asymptotic behavior of expected bucket pair counts $\mathrm{E}[\binom{N_{i,n}}{2}]$ also has ties to the clustering coefficient for the configuration model [14]. This is usually defined as a (perhaps scaled) ratio between the number of triangles in a graph and the number of paths of length 2 (partial triangles). For many reference degree distributions, such as Poisson (where all moments are finite), the clustering coefficient tends to 0 at rate $O(n^{-1})$ [20]. However, for highly skewed degree distributions, the convergence of the clustering coefficient depends more closely on the moments of the distribution. In particular, power law degree distributions require an index $\alpha > 2\frac{1}{3}$ for the clustering coefficient to converge to zero [14], which is exactly the power index required for expected number of pairs for a bucket to converge finitely.

3 Proof of the Main Theorem

In this section, we give the proof for Theorem 1. Recall that producing a graph under the erased configuration model (ECM) first requires generating a configuration model (CM) graph based on stubs drawn from an truncated reference distribution. To study ECM graphs, we must usually formulate and examine random events which may occur under the CM before any erasure for the ECM. For example, in order for nodes to be neighbors in an ECM graph, the nodes must first be neighbors in a pre-erasure CM graph. We will make such events clear in the following.

If A and B represent two generic events, let "A, B" denote their set intersection and define the indicator function of an event A as $\mathbb{I}(A) = 1$ if A holds and otherwise $\mathbb{I}(A) = 0$. In a graph containing n nodes, recall $D_{1,n}, \ldots, D_{n,n}$ are pre-erasure stubs for nodes drawn independently from (3) and $D^s_{1,n}, \ldots, D^s_{n,n}$ denote final degrees in the simple graph under the ECM.

With the ECM generation model, in a graph with n nodes, the distribution of a bucket size $N_{i,n}$ does not depend on the particular node i. WLOG we pick and fix node 1, noting that $\mathrm{E}[N_{i,n}] = \mathrm{E}[N_{1,n}]$ and $\mathrm{E}[\binom{N_{i,n}}{2}] = \mathrm{E}[\binom{N_{1,n}}{2}]$ for $1 \le i \le n$. Let $A_{1,j,n}$ denote the event that a node $j \neq 1$ is a neighbor of a node 1 in the ECM and that $D^s_{j,n} \ge D^s_{1,n}$ holds. Thus edge $(1, j)$ will be in node 1's bucket in Cohen's algorithm. The number of elements in node 1's bucket $N_{1,n} = \sum_{j=2}^{n} \mathbb{I}(A_{1,j,n})$. Thus,

$$\mathrm{E}[N_{1,n}] = \sum_{j=2}^{n} \mathrm{E}\big[\mathbb{I}(A_{1,j,n})\big] = (n-1)P(A_{1,2,n}),$$

since $\mathrm{E}\big[\mathbb{I}(A_{1,j,n})\big] = P(A_{1,j,n}) = P(A_{1,2,n})$ for $2 \le j \le n$. Similarly,

$$\begin{aligned} \mathrm{E}[N_{1,n}^2] &= \sum_{j=1, j\neq 1}^{n} \sum_{k=1, k\neq 1}^{n} \mathrm{E}\big[\mathbb{I}(A_{1,j,n}, A_{1,k,n})\big] \\ &= (n-1)(n-2)P(A_{1,2,n}, A_{1,3,n}) + (n-1)P(A_{1,2,n}) \end{aligned}$$

because $\mathbb{I}(A_{1,j,n}, A_{1,k,n}) = \mathbb{I}(A_{1,j,n})$ if $j = k$ and $\mathrm{E}\big[\mathbb{I}(A_{1,j,n}, A_{1,k,n})\big] = P(A_{1,j,n}, A_{1,k,n}) = P(A_{1,2,n}, A_{1,3,n})$ for any distinct nodes $1 < j, k \le n$. Substitution of these expressions yields

$$\mathrm{E}\left[\binom{N_{1,n}}{2}\right] = \frac{1}{2}(\mathrm{E}[N_{1,n}^2] - \mathrm{E}[N_{1,n}]) = \binom{n-1}{2} P(A_{1,2,n}, A_{1,3,n}).$$

To consolidate notation, for fixed $m \in \{2, 3\}$, let $\mathrm{NB}_{m,n}$ denote the event that node 1 is a neighbor of each node from node 2 through node m in the ECM (or equivalently in the pre-erasure CM) and let $\mathrm{NBD}_{m,n}$ denote the compound event that $\mathrm{NB}_{m,n}$ holds along with $D_{i,n}^s \ge D_{1,n}^s$ for $i = 1, \ldots, m$. We have

$$\mathrm{E}[N_{1,n}] = (n-1)P(\mathrm{NBD}_{2,n}) \qquad \mathrm{E}[\binom{N_{1,n}}{2}] = \binom{n-1}{2} P(\mathrm{NBD}_{3,n}).$$

Theorem 1 will follow immediately by showing that

$$\lim_{n\to\infty} n^{m-1} P(\mathrm{NBD}_{m,n}) = \begin{cases} \dfrac{1}{\mathrm{E}[D]} \displaystyle\sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} d_1 d_2 f(d_1) f(d_2) & \text{if } m = 2 \\ \dfrac{1}{\mathrm{E}[D]^2} \displaystyle\sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} d_1(d_1-1) d_2 d_3 f(d_1) f(d_2) f(d_3) & \text{if } m = 3, \end{cases} \tag{4}$$

under moment conditions on the degree distribution (1) (either $\mathrm{E}[D] = O(1)$ for the $m = 2$ case or $\mathrm{E}[D^{4/3}] = O(1)$ for the $m = 3$ case).

To show (4), we fix $m \in \{2, 3\}$ and expand the probability $P(\mathrm{NBD}_{m,n})$ in terms of the pre-erasure stubs $D_{1,n}, \ldots, D_{m,n}$. For $m = 2, 3$, let $\mathrm{SLM}_{m,n}$ denote the event that, before erasing, some node i among $\{1, \ldots, m\}$ has at least one self-loop or at least one multi-edge or that the stubs $D_{i,n}$ of node i have been incremented by 1. Recall that when $\sum_{i=1}^{n} D_{i,n}$ is odd, the stubs of one randomly selected node are incremented by 1 before random pairing of stubs. Then, the complement $\mathrm{SLM}_{m,n}^c$ is the event that no nodes among $\{1, \ldots, m\}$ have any self-loops or multi-edges in the pre-erasure CM (i.e., before erasing in the ECM) and that no nodes among $\{1, \ldots, m\}$ have had their initial draws of stubs incremented. For a generic event B_n, let $P(B_n | d_1, \ldots, d_m)$ denote the condition probability of B_n given realized values $D_{1,n} = d_1, \ldots, D_{m,n} = d_m$ of $D_{1,n}, \ldots, D_{m,n}$. The following probability of intersecting events may be written conditionally,

$$P(B_n, D_{1,n} = d_1, \ldots, D_{m,n} = d_m) = P(B_n | d_1, \ldots, d_m) P(D_{1,n} = d_1, \ldots, D_{m,n} = d_m).$$

We have $P(D_{1,n} = d_1, \ldots, D_{m,n} = d_m) = \prod_{i=1}^{m} f_n(d_i)$ using (3) and independence. We then may decompose $P(\mathrm{NBD}_{m,n})$ as a sum of probabilities $P(\mathrm{NBD}_{m,n}, D_{1,n} = d_1, \ldots, D_{m,n} = d_m)$ over every possible realization $(d_1, \dots, d_m)$ of pre-erasure stub draws $(D_{1,n}, \dots, D_{m,n})$:

$$
\begin{aligned}
P(\mathrm{NBD}_{m,n}) &= \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} P(\mathrm{NBD}_{m,n}, D_{1,n} = d_1, \dots, D_{m,n} = d_m) \\
&= \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} P(\mathrm{NBD}_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i) \\
&= \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} P(\mathrm{NBD}_{m,n}, \mathrm{SLM}_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i) + \\
&\quad \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} P(\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i),
\end{aligned} \tag{5}
$$

where the last line above follows from decomposing $P(\mathrm{NBD}_{m,n}|d_1, \dots, d_m)$ as the sum of conditional probabilities for disjoint events $\mathrm{NBD}_{m,n}, \mathrm{SLM}_{m,n}$ and $\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n}$.

When $\mathrm{SLM}^c_{m,n}$ holds, it follows that $D_{i,n} = D^s_{i,n}$ for $i = 1, \dots, m$ so that $D^s_{i,n} \geq D^s_{1,n}$ becomes equivalent to $D_{i,n} \geq D_{1,n}$ for $i = 1, \dots, m$. Consequently, it follows that

$$
P(\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) = \begin{cases} P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) & \text{if } d_2, \dots, d_m \geq d_1 \\ 0 & \text{otherwise.} \end{cases} \tag{6}
$$

This means that, conditional on $D_{1,n} = d_1, \dots, D_{m,n} = d_m$, if $d_i \geq d_1$ holds for every $i \in \{1, \dots, m\}$ then events "$\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n}$" and "$\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}$" are equivalent. That is, if $\mathrm{SLM}^c_{m,n}$ already holds given $D_{1,n} = d_1, \dots, D_{m,n} = d_m$ with $d_i \geq d_1$, then so does $D^s_{i,n} = d_i \geq D^s_{1,n} = d_1$ for $i = 1, \dots, m$. Therefore, event $\mathrm{NBD}_{m,n}$ may hold in addition to $\mathrm{SLM}^c_{m,n}$ if and only if $\mathrm{NB}_{m,n}$ holds additionally. Also given $d_i < d_1$ holds for some $i \in \{1, \dots, m\}$, $\mathrm{NBD}_{m,n}$ and $\mathrm{SLM}^c_{m,n}$ cannot occur simultaneously and so $P(\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) = 0$ in (6). For fixed $m = 2, 3$, we define a scaled probability based on (5) and (6) as

$$
\begin{aligned}
p^{(1)}_{m,n} &\equiv n^{m-1} \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} P(\mathrm{NBD}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i) \\
&= \sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} \cdots \sum_{d_m=d_1}^{\infty} n^{m-1} P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i)
\end{aligned}
$$

(where summation indices $d_2, \dots, d_m$ range from d_1 to ∞ in the second line above by (6)). Using this with (5), we may bound

$$
p^{(1)}_{m,n} \leq n^{m-1} P(\mathrm{NBD}_{m,n}) \leq p^{(1)}_{m,n} + p^{(2)}_{m,n}
$$

where

$$
p^{(2)}_{m,n} \equiv \sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} n^{m-1} P(\mathrm{NBD}_{m,n}, \mathrm{SLM}_{m,n} | d_1, \dots, d_m) \prod_{i=1}^{m} f_n(d_i).
$$

Now to show (4) for $m = 2, 3$ and thereby Theorem 1, it suffices to establish

$$
\begin{aligned}
\lim_{n \to \infty} p^{(1)}_{m,n} &= \begin{cases} \dfrac{1}{\mathrm{E}[D]} \displaystyle\sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} d_1 d_2 f(d_1) f(d_2) & \text{if } m = 2 \\ \dfrac{1}{[\mathrm{E}[D]]^2} \displaystyle\sum_{d_1=0}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} d_1(d_1 - 1) d_2 d_3 f(d_1) f(d_2) f(d_3) & \text{if } m = 3, \end{cases} \\
\lim_{n \to \infty} p^{(2)}_{m,n} &= 0.
\end{aligned} \tag{7}
$$

To prove (7), we use the Lesbegue Dominated Convergence Theorem (LDCT) along with technical results in Lemma 1 below (which will be proven in Section B). Recall that $P(\mathrm{NB}_{m,n}|d_1,\ldots,d_m)$ denotes the conditional probability that node 1 is a neighbor of nodes 2 through m (in the pre-erasure CM or in the ECM) given the observed stubs $D_{1,n}=d_1,\ldots,D_{m,n}=d_m$.

Lemma 1. *Under the CM (i.e., pre-erasure ECM) for generating random graphs (involving truncation of the reference degree distribution (1)), suppose* $\mathrm{E}[D]=\sum_{d=0}^{\infty} d\cdot f(d)\in(0,\infty)$*. Fix* $m=2$ *or* 3.
(i) Then, for any arbitrary integers $d_1,\ldots,d_m\geq 0$ *with* $\prod_{i=1}^{m} f(d_i)>0$,

$$\begin{aligned}&\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m) \\ =\;&\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n}|d_1,\ldots,d_m) \\ =\;&\begin{cases} d_1d_2/\mathrm{E}[D] & \text{if } m=2 \\ d_1(d_1-1)d_2d_3/(\mathrm{E}[D])^2 & \text{if } m=3.\end{cases}\end{aligned} \tag{8}$$

(ii) For $m=2$*, there exists a real* $C>0$ *such that*

$$0\leq nP(\mathrm{NBD}_{2,n},\mathrm{SLM}_{2,n}|d_1,d_2),\, nP(\mathrm{NB}_{2,n},\mathrm{SLM}^c_{2,n}|d_1,d_2)\leq nP(\mathrm{NB}_{2,n}|d_1,d_2)\leq Cd_1d_2$$

holds for any $n\geq 3$ *and all integers* $d_1,d_2\geq 0$.
(iii) For $m=3$*, there exists a real* $C>0$ *such that*

$$0\leq n^2P(\mathrm{NBD}_{3,n},\mathrm{SLM}_{3,n}|d_1,d_2,d_3)\prod_{i=1}^{3} f_n(d_i)\leq C(d_1d_2d_3)^{4/3}\prod_{i=1}^{3} f(d_i)$$

holds for any $n\geq 4$ *and all integers* $d_1,d_2,d_3\geq 0$ *and*

$$0\leq n^2P(\mathrm{NB}_{3,n},\mathrm{SLM}^c_{3,n}|d_1,d_2,d_3)\leq n^2P(\mathrm{NB}_{3,n}|d_1,d_2,d_3)\leq C(d_1d_2d_3)^{4/3}$$

holds for any $n\geq 4$ *and all integers* $d_2,d_3\geq d_1\geq 0$.

The LDCT allows limits (7) of sums $p^{(1)}_{m,n},p^{(2)}_{m,n}$ (i.e., sums over integer-tuples $(d_1,\ldots,d_m)$) to be determined by sums of limits

$$\begin{aligned}\lim_{n\to\infty} p^{(1)}_{m,n} &= \sum_{d_1=0}^{\infty}\sum_{d_2=d_1}^{\infty}\cdots\sum_{d_m=d_1}^{\infty}\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m)\prod_{i=1}^{m} f_n(d_i) \\ \lim_{n\to\infty} p^{(2)}_{m,n} &= \sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}_{m,n}|d_1,\ldots,d_m)\prod_{i=1}^{m} f_n(d_i).\end{aligned} \tag{9}$$

Assuming that (9) indeed holds, the substitution of $\lim_{n\to\infty}\prod_{i=1}^{m} f_n(d_i)=\prod_{i=1}^{m} f(d_i)$ and limiting values of $\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m)$ from Lemma 1(i) into (9) (for given values $d_1,\ldots,d_m\geq 0$) will establish the $\lim_{n\to\infty} p^{(1)}_{m,n}$ result in (7) and substitution of

$$\lim_{n\to\infty} n^{m-1}P(\mathrm{NBD}_{m,n},\mathrm{SLM}_{m,n}|d_1,\ldots,d_m)=0 \tag{10}$$

in (9) will establish $\lim_{n\to\infty} p^{(2)}_{m,n}=0$ in (7), where

$$\begin{aligned}0 &= \lim_{n\to\infty} n^{m-1}\big[P(\mathrm{NB}_{m,n}|d_1,\ldots,d_m)-P(\mathrm{NB}_{m,n},\mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m)\big] \\ &= \lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}_{m,n}|d_1,\ldots,d_m) \\ &\geq \lim_{n\to\infty} n^{m-1}P(\mathrm{NBD}_{m,n},\mathrm{SLM}_{m,n}|d_1,\ldots,d_m)\geq 0\end{aligned}$$

follows from Lemma 1(i) and $P(\mathrm{NB}_{m,n}, \mathrm{SLM}_{m,n}|d_1,\ldots,d_m) \geq P(\mathrm{NBD}_{m,n}, \mathrm{SLM}_{m,n}|d_1,\ldots,d_m)$ by $\mathrm{NBD}_{m,n} \subseteq \mathrm{NB}_{m,n}$. The proof of Theorem 1 will then be finished after justifying (9).

For fixed $m = 2, 3$, the order of summation and limits may be validly exchanged in (9) under the LDCT because (under the assumptions of Theorem 1) there exists a “dominating" nonnegative real-valued function $g_m(d_1,\ldots,d_m)$ of integers $d_1,\ldots,d_m \geq 0$ (depending on $m = 2, 3$ but not n) for which it holds that

1. $n^{m-1} P(\mathrm{NBD}_{m,n}, \mathrm{SLM}_{m,n}|d_1,\ldots,d_m) \prod_{i=1}^m f_n(d_i) \leq g_m(d_1,\ldots,d_m)$ for any $n \geq 4$, all integers $d_1,\ldots,d_m \geq 0$;
2. $n^{m-1} P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m) \prod_{i=1}^m f_n(d_i) \leq g_m(d_1,\ldots,d_m)$ for all $n \geq 4$, all integers $d_2,\ldots,d_m \geq d_1 \geq 0$;
3. and
$$\sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} g_m(d_1,\ldots,d_m) < \infty. \tag{11}$$

In other words, for any $n \geq 4$, $g_m(d_1,\ldots,d_m)$ dominates each summand of $p^{(1)}_{m,n}$ over the range of summation defining $p^{(1)}_{m,n}$ (i.e., integers $d_2,\ldots,d_m \geq d_1 \geq 0$) and dominates each summand of $p^{(2)}_{m,n}$ over the range of summation defining $p^{(2)}_{m,n}$ (i.e., integers $d_1,\ldots,d_m \geq 0$), while by (11) $g_m(d_1,\ldots,d_m)$ is finitely summable over all integers $d_1,\ldots,d_m \geq 0$ (the largest possible range of summation). Hence, the order of limits and summation may be exchanged in (9) via the LDCT (cf. Ch. 3, Athreya and Lahiri, 2006). The dominating function $g_m(d_1,\ldots,d_m)$ is defined as follows. From Lemma 1(ii), we may set $g_2(d_1,d_2) = C d_1 d_2 f(d_1) f(d_2)$ for some real $C > \sup_{n\geq 1} C_n^{-2} > 0$ when $m = 2$ so that a finite sum in (11) follows by $\mathrm{E}[D] < \infty$ in this case under Theorem 1 (note from (3) that $f_n(d) \leq f(d) \sup_{n\geq 1} C_n^{-2}$ for all $n \geq 1$, $d \geq 0$); under the additional assumption in Theorem 1(ii) that $\mathrm{E}[D^{4/3}] = O(1)$, we use Lemma 1(iii) to define $g_3(d_1,d_2,d_3) = C \prod_{i=1}^3 f(d_i) d_i^{4/3}$ for some real $C > 0$ when $m = 3$ so that a finite sum in (11) follows by $\mathrm{E}[D^{4/3}] = O(1)$ in this case. □

Remark 2: The proof above has some implications for probabilities of neighbor relationships in the CM or in the ECM. We mention two conditional probabilities (i.e., $d_2 d_1/[n\mathrm{E}[D]]$, $d_1(d-1)d_2 d_3/[n\mathrm{E}[D]]^2$) which asymptotically contribute to the sums in Theorem 1.

The limit (10) shows that, asymptotically, the possibility of self-loops or multi-edges involving at least one of the nodes among $\{1,\ldots,m\}$ is negligible in determining the conditional probability that node 1 is a neighbor of nodes 2 through m given $D_{1,n} = d_1,\ldots,D_{m,n} = d_m$. In other words, (8) says that when $m = 2$ and we consider the conditional probability that nodes 1 and 2 are neighbors in a large CM graph,

$$P(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n}|d_1,d_2) \approx P(\mathrm{NB}_{2,n}|d_1,d_2) \approx d_2 \frac{d_1}{n\mathrm{E}[D]}$$

holds with the interpretation that (in a large graph and given $D_{1,n} = d_1, D_{2,n} = d_2$) the conditional probability $d_2 d_1/[n\mathrm{E}[D]]$ that nodes 1 and 2 are neighbors in a graph from the CM (or even from the ECM) is the same as the conditional probability that the pre-erasure CM yields a graph where nodes 1 and 2 are neighbors but with no self-loops or multi-edges involving these nodes. In a sense, we pick exactly one stub from d_2 stubs available for node 2 and then connect it to a stub of node 1 with approximate probability $d_1/n\mathrm{E}[D]$ in random rewiring (where there are n nodes in the graph with $\mathrm{E}[D] = \sum_{d=0}^{\infty} d \cdot f(d)$ stubs on average), so that $d_2 d_1/[n\mathrm{E}[D]]$ represents the probability that nodes 1 and 2 are neighbors (with or without any multi-edges or self-loops) in a CM graph.

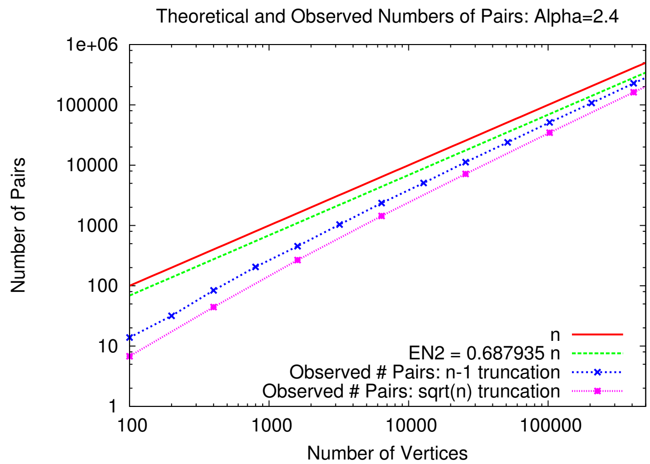

Figure 1: Experimental results with the truncated and non-truncated ECM. The red (solid) line shows the number of nodes n. The green dashed, plain line shows the computed bound on the expected number of pairs per bucket. The green (dashed, crosses) line shows the number of observed pairs per bucket in our experiments when truncating the degree at $n-1$, and the magenta (dashed, boxes) line shows the number of observed pairs per bucket when truncating the degree at $\sqrt{n}$.

When $m = 3$, the conditional probability that node 1 is a neighbor of nodes 2 and 3 given $D_{1,n} = d_1, D_{2,n} = d_2, D_{3,n} = d_3$ in a CM large graph is

$$P(\mathrm{NB}_{3,n}, \mathrm{SLM}^c_{3,n} | d_1, d_2, d_3) \approx P(\mathrm{NB}_{3,n} | d_1, d_2, d_3) \approx d_2 \frac{d_1}{n\mathrm{E}[D]} \times d_3 \frac{d_1 - 1}{n\mathrm{E}[D]},$$

where, after connecting one stub from node 1 to one stub from node 2 with probability $d_2 d_1/[n\mathrm{E}[D]]$, we pick one of the d_3 stubs of node 3 to connect to a remaining stub of node 1 with probability $(d_1 - 1)/n\mathrm{E}[D]$.

4 Experimental Results

We have proved that the expected number of pairs in Cohen's buckets grows linearly in the truncated ECM when the reference degree distribution has a finite 4/3 moment. Power Law graphs with $\alpha = 2.4$, and with degree truncation at $n^{0,\frac{1}{2}}$ have this moment. In real power law graph instances, however, there will be vertices of degree greater than $n^{\frac{1}{2}}$. We have shown in Corollary 1 that the number of pairs in a bucket in that case grows super-linearly. However, we now offer empirical evidence that the effect of truncating at $n-1$, i.e., not truncating, is small. Figure 1 shows that when $\alpha = 2.4$, the average number of pairs observed in 100 Monte Carlo trials at each of various ECM graph sizes is very close to the average number with observed with $\sqrt{n}$ truncation. In fact, as graph sizes increase past several million vertices, we still observe fewer than one pair per bucket on average in our experiments with the $n-1$ truncated ECM. Thus, even in a regime in which we cannot prove a linear time bound for Cohen's algorithm, our experiments indicate that its true running time is a slow-growing function of n.

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B Appendix: Proof of Lemma 1

To establish Lemma 1, we require some additional notation and a technical result presented in Lemma 2 below.

In constructing a generic graph under the pre-erasure CM, suppose that, before any random wiring, that a given node A_1 as a_1 stubs, a second given node A_2 has a_2 stubs, and that s is the combined number of stubs among all other nodes in the graph, where $a_1, a_2, s \geq 0$ are integers with $a_1 + a_2 + s \geq 2$ even. The total number of configurations which pair $a_1 + a_2 + s$ stubs into $(a_1 + a_2 + s)/2$ edges in a CM graph is given by

$$\frac{1}{[(a_1+a_2+s)/2]!}\binom{a_1+a_2+s}{2}\binom{a_1+a_2+s-2}{2}\cdots\binom{2}{2} = P^{a_1+a_2+s}_{(a_1+a_2+s)/2}2^{-(a_1+a_2+s)/2},$$

expressed using the permutation function $P^y_x = y!/(y-x)!$ for integers $y \geq x \geq 0$ and $P^y_x = 0$ if $x > y \geq 0$. (Above the factor $[(a_1 + a_2 + s)/2]!$ adjusts for the fact that the order in which $(a_1 + a_2 + s)/2$ pairs are formed is irrelevant.) Then, given values of a_1, a_2, s, the probability that the pre-erasure CM produces a graph where node A_1 has exactly $0 \leq k \leq \lfloor a_1/2 \rfloor$ self-loops and shares no edges with node A_2 is given by a function

$$\begin{aligned} & h(a_1, a_2, s, k) \\ \equiv\ & \frac{1}{k!}\binom{a_1}{2}\cdots\binom{a_1-2k}{2}P^s_{a_1-2k}P^{s+a_2-(a_1-2k)}_{[s+a_2-(a_1-2k)]/2}2^{-[s+a_2-(a_1-2k)]/2}\Big/P^{a_1+a_2+s}_{(a_1+a_2+s)/2}2^{-(a_1+a_2+s)/2} \\ =\ & \frac{2^{a_1-2k}}{k!}P^{a_1}_{2k}P^s_{a_1-2k}P^{s+a_2-(a_1-2k)}_{[s+a_2-(a_1-2k)]/2}\Big/P^{a_1+a_2+s}_{(a_1+a_2+s)/2}, \end{aligned}$$

where, in the numerator above, $\frac{1}{k!}\binom{a_1}{2}\cdots\binom{a_1-2k}{2}$ is the number of ways to pick and pair $2k$ stubs from a_1 stubs to form k self-loops, $P^s_{a_1-2k}$ is the number of ways to pair the remaining $a_1 - 2k$ stubs of node A_1 to the collection of s stubs (not involving stubs of node A_2), and $P^s_{a_1-2k}P^{s+a_2-(a_1-2k)}_{[s+a_2-(a_1-2k)]/2}2^{-[s+a_2-(a_1-2k)]/2}$ is the number of ways to pair the remaining $s+a_2-(a_1-2k)$

stubs in the graph. Accounting for any potential number of self-loops involving node A_1, the probability of a CM graph where node A_1 shares no edges with node A_2 is given by the sum of probabilities of disjoint events

$$\sum_{k=0}^{\lfloor a_1/2 \rfloor} h(a_1, a_2, s, k)$$

and the conditional probability that node A_1 and A_2 are neighbors (share at least one edge) in a CM graph is then

$$p_1(a_1, a_2, s) \equiv 1 - \sum_{k=0}^{\lfloor a_1/2 \rfloor} h(a_1, a_2, s, k), \tag{12}$$

given values of a_1, a_2, s.

Additionally, using (12), we may derive one other conditional probability of interest in a CM graph. Before any random wiring in the pre-erasure CM, suppose that three nodes A_1, A_2, A_3 have given values of stubs a_1, a_2, a_3 and that s is the combined number of stubs among all other nodes in the graph, where $a_1, a_2, a_3, s \geq 0$ are integers with $a_1 + a_2 + a_3 + s \geq 2$ even. Then, the conditional probability that node A_1 is a neighbor of nodes A_2 and A_3 in a CM graph is given by

$$p_2(a_1, a_2, a_3, s) \equiv p_1(a_1, a_2, a_3 + s) + p_1(a_1, a_3, a_2 + s) - p_1(a_1, a_2 + a_3, s) \tag{13}$$

using the inclusion-exclusion law and the conditional probability function (12); in other words, the (conditional) probability $p_1(a_1, a_2 + a_3, s)$ that node A_1 shares at least one edge with node A_2 or node A_3 is “the probability $p_1(a_1, a_2, a_3 + s)$ that node A_1 shares at least one edge with node A_2 plus the probability $p_1(a_1, a_3, a_2 + s)$ that node A_1 shares at least one edge with node A_3 minus the probability $p_2(a_1, a_2, a_3, s)$ that node A_1 shares at least one edge with both nodes A_2 and A_3."

We now summarize some limiting behavior of the conditional probabilities $p_1(a_1, a_2, s)$ and $p_2(a_1, a_2, a_3, s)$ of neighboring relationships, given stub counts a_1, a_2, s, when the number of “remaining" stubs in the graph $s \to \infty$.

Lemma 2. *Fix integers $a_1, a_2, a_3 \geq 0$.*
(i) For any integer $s \geq 0$, let $s^* = s$ if $a_1 + a_2 + s$ is even and, otherwise, let $s^* = s + 1$. Then,

$$\lim_{s \to \infty} s p_1(a_1, a_2, s^*) = a_1 a_2$$

(ii) For any integer $s \geq 0$, let $s^* = s$ if $a_1 + a_2 + a_3 + s$ is even and, otherwise, let $s^* = s + 1$. Then,

$$\lim_{s \to \infty} s^2 p_2(a_1, a_2, a_3, s^*) = a_1(a_1 - 1) a_2 a_3.$$

We defer the proof of Lemma 2 to Section B.0.5 and consider now establishing Lemma 1.

To simplify the exposition, we shall first give a prove of Lemma 1 *under the additional assumption that* $f(0) = 0$ in the reference distribution (1). This implies that, for any $n \geq 1$, the initial stub size $D_{i,n}$ of any node $i \in \{1, \ldots, n\}$ must be at least 1. The resulting proof of Lemma 1 may then be modified to treat the possibility that $f(0) > 0$; we describe the modification in Section B.0.4.

B.0.1 Proof of Lemma 1(i)

We require an expansion of the conditional probability $P(\mathrm{NB}_{m,n} | d_1, \ldots, d_m)$ that node 1 is a neighbor of nodes 2 through m in the pre-erasure CM (or in the ECM), for fixed $m = 2, 3$.

Let $S_{n-m} \equiv \sum_{i=m+1}^{n} D_{i,n}$ represent the partial sum of all (pre-erasure) stubs excluding nodes 1 through m, let $f_{S_{n-m}}(s) = P(S_{n-m} = s)$, $s \geq n-m$ denote the probability function of S_{n-m}, and let $P(\text{NB}_{m,n}|d_1,\ldots,d_m,s) \equiv P(\text{NB}_{m,n}, D_{1,n} = d_1,\ldots,D_{m,n} = d_m, S_{n-m} = s)/P(D_{1,n} = d_1,\ldots,D_{m,n} = d_m, S_{n-m} = s)$ denote the conditional probability that node 1 is a neighbor of nodes 2 through m in the pre-erasure CM given that $D_{1,n} = d_1,\ldots,D_{m,n} = d_m, S_{n-m} = s$. Then,

$$\begin{aligned}
P(\text{NB}_{m,n}|d_1,\ldots,d_m) &= \frac{P(\text{NB}_{m,n}, D_{1,n} = d_1,\ldots,D_{m,n} = d_m)}{P(D_{1,n} = d_1,\ldots,D_{m,n} = d_m)} \\
&= \sum_{s=n-m}^{\infty} \frac{f_{S_{n-m}}(s)}{f_{S_{n-m}}(s)} \frac{P(\text{NB}_{m,n}, D_{1,n} = d_1,\ldots,D_{m,n} = d_m, S_{n-m} = s)}{P(D_{1,n} = d_1,\ldots,D_{m,n} = d_m)} \\
&= \sum_{s=n-m}^{\infty} f_{S_{n-m}}(s) P(\text{NB}_{m,n}|d_1,\ldots,d_m,s) \qquad (14)
\end{aligned}$$

using $f_{S_{n-m}}(s)P(D_{1,n} = d_1,\ldots,D_{m,n} = d_m) = P(D_{1,n} = d_1,\ldots,D_{m,n} = d_m, S_{n-m} = s)$ by the independence of $D_{1,n},\ldots,D_{n,n}$. We may now determine a further expression for $P(\text{NB}_{m,n}|d_1,\ldots,d_m,s)$ using the conditional probability functions developed in (12) and (13). For $m = 2$, the conditional probability $P(\text{NB}_{2,n}|d_1,d_2,s)$ that node 1 is a neighbor of node 2 given $D_{1,n} = d_1, D_{2,n} = d_2, S_{n-2} = s$ is then given by

$$\begin{aligned}
&P(\text{NB}_{2,n}|d_1,d_2,s) \qquad (15)\\
&= \begin{cases} p_1(d_1,d_2,s) & \text{even } d_1+d_2+s \\ \frac{1}{n}p_1(d_1+1,d_2,s) + \frac{1}{n}p_1(d_1,d_2+1,s) + \frac{n-2}{n}p_1(d_1,d_2,s+1) & \text{odd } d_1+d_2+s, \end{cases}
\end{aligned}$$

where, in the case of odd $d_1 + d_2 + s$, one of the n nodes is randomly picked and its stub count is incremented by 1. For $m = 3$, the conditional probability $P(\text{NB}_{3,n}|d_1,d_2,d_3,s)$ that node 1 is a neighbor of nodes 2 and 3 given $D_{1,n} = d_1, D_{2,n} = d_2, D_{3,n} = d_3, S_{n-3} = s$ is then

$$\begin{aligned}
&P(\text{NB}_{3,n}|d_1,d_2,d_3,s) \qquad (16)\\
&= \begin{cases} p_2(d_1,d_2,d_3,s) & \text{even } d_1+d_2+d_3+s \\ \frac{1}{n}p_2(d_1+1,d_2,d_3,s) + \frac{1}{n}p_2(d_1,d_2+1,d_3,s) & \text{odd } d_1+d_2+d_3+s \\ \quad +\frac{1}{n}p_2(d_1,d_2,d_3+1,s) + \frac{n-3}{n}p_2(d_1,d_2,d_3,s+1) & \end{cases}
\end{aligned}$$

To show $\lim_{n\to\infty} nP(\text{NB}_{2,n}|d_1,d_2) = d_1d_2/\text{E}[D]$ in Lemma 1(i) for $m = 2$, use (14) to write

$$\begin{aligned}
&\left| nP(\text{NB}_{2,n}|d_1,d_2) - \frac{d_1d_2}{\text{E}[D]} \right| \qquad (17)\\
&= \left| \sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s) \left[sP(\text{NB}_{2,n}|d_1,d_2,s) - d_1d_2 \right] + d_1d_2 \left(\sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s) - \frac{1}{\text{E}[D]} \right) \right| \\
&\leq \frac{n}{n-2} \sup_{s\geq n-2} |sP(\text{NB}_{2,n}|d_1,d_2,s) - d_1d_2| + d_1d_2 \left| \text{E}\left[\frac{n}{S_{n-2}}\right] - \frac{1}{\text{E}[D]} \right|
\end{aligned}$$

using above

$$\text{E}\left[\frac{n}{S_{n-2}}\right] = \sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s) \leq \frac{n}{n-2} \sum_{s=n-2}^{\infty} f_{S_{n-2}}(s) = \frac{n}{n-2}.$$

By Lemma 2(i) and (15), it holds that

$$\lim_{n\to\infty} \sup_{s\geq n-2} \left| sP(\text{NB}_{2,n}|d_1,d_2,s) - \frac{d_1d_2}{\text{E}[D]} \right| = 0.$$

We next show $\mathrm{E}[(n/S_{n-2})] \to 1/\mathrm{E}[D]$, which by (17) then proves $\lim_{n\to\infty} nP(\mathrm{NB}_{2,n}|d_1,d_2,s) = d_1d_2/\mathrm{E}[D]$ in Lemma 1(i) for $m=2$. (Note that, for any slowly varying function $L(n)$, it holds that $\lim_{n\to\infty} n^y L(n) = \infty$ and $\lim_{n\to\infty} n^{-y}L(n) = 0$ for any $y>0$, which are used below.) Since $C_n\mathrm{E}[D_{1,n}] = \sum_{d=0}^{n^\tau L(n)} d\cdot f(d) \le \mathrm{E}[D] = O(1)$ for $n \ge 1$ and $\lim_{n\to\infty} C_n = 1$, it follows that $\lim_{n\to\infty} \mathrm{E}[S_{n-2}/n] = \lim_{n\to\infty} \mathrm{E}[D_{1,n}](n-2)/n = \mathrm{E}[D]$. Also, because $D_{1,n},\ldots,D_{n,n}$ are independently, identically distributed and bounded by $n^\tau L(n)$, it follows that

$$\mathrm{E}\left[\frac{S_{n-2}}{n} - \mathrm{E}[\frac{S_{n-2}}{n}]\right]^2 = \frac{n-2}{n^2}\mathrm{E}[D_{1,n}]^2 \le \frac{n-2}{n^2}[L(n)n^\tau]^2 \to 0$$

as $n\to\infty$ (since $\tau\in(0,1/2)$ and $L(\cdot)$ is slowly varying in (3)), which then implies $S_{n-2}/n \to \mathrm{E}[D]$ in distribution as $n\to\infty$. By Skorohod's embedding theorem (cf. Athreya and Lahiri, 2006, ch. 8), there exist random variables Y_n, $n\ge 3$, defined on a common probability space, such that Y_n has the same distribution as S_{n-2}/n for each n and $Y_n \to \mathrm{E}[D] > 0$ with probability 1 (w.p.1) as $n\to\infty$. Then, as $n\to\infty$, $1/Y_n \to 1/\mathrm{E}[D] > 0$ w.p.1 and $1/Y_n \le n/[n-2] \le 3$ is bounded w.p.1 for all $n\ge 3$ so that $\mathrm{E}[(n/S_{n-2})] = \mathrm{E}[(1/Y_n)] \to 1/\mathrm{E}[D]$ by the LDCT.

To prove $\lim_{n\to\infty} n^2 P(\mathrm{NB}_{3,n}|d_1,d_2,d_3) = d_1(d_1-1)d_2d_3/[\mathrm{E}[D]]^2$ in Lemma 1(i) for $m=3$, use (14) and (16) to write

$$P(\mathrm{NB}_{3,n}|d_1,d_2,d_3) = \sum_{s=n-3}^{\infty} f_{S_{n-2}}(s)p_2(d_1,d_2,d_3,s^*) + \frac{1}{n}\sum_{s\in\mathcal{S}_n} f_{S_{n-2}}(s)\tilde{p}_2(d_1,d_2,d_3,s)$$

where $s^* = s$ if $d_1+d_2+d_3+s$ is even and, otherwise, $s^* = s+1$ and $\mathcal{S}_n \equiv \{s \ge n-2 : d_1+d_2+d_3+s \text{ odd}\}$ with

$$\tilde{p}_2(d_1,d_2,d_3,s) \equiv p_2(d_1+1,d_2,d_3,s)+p_2(d_1,d_2+1,d_3,s)+p_2(d_1,d_2,d_3+1,s)-3p_2(d_1,d_2,d_3,s+1).$$

Then,

$$\left|n^2 P(\mathrm{NB}_{3,n}|d_1,d_2,d_3) - \frac{d_1(d-1)d_2d_3}{(\mathrm{E}[D])^2}\right| \le I_{1n}+I_{2n}+I_{3n}$$

where

$$\begin{aligned}
I_{1n} &\equiv \left|\sum_{s=n-3}^{\infty}\frac{n^2}{s^2}f_{S_{n-2}}(s)\left[s^2p_2(d_1,d_2,d_3,s^*) - d_1(d-1)d_2d_3\right]\right| \\
&\le \left(\frac{n}{n-3}\right)^2 \sup_{s\ge n-3}\left|s^2p_2(d_1,d_2,d_3,s^*) - d_1(d-1)d_2d_3\right|, \\
I_{2n} &\equiv \sum_{s\in\mathcal{S}_n}\frac{n}{s}f_{S_{n-2}}(s)s\tilde{p}_2(d_1,d_2,d_3,s) \;\le\; \frac{n}{n-3}\sup_{s\in\mathcal{S}_n} s\tilde{p}_2(d_1,d_2,d_3,s), \\
I_{3n} &\equiv d_1(d_1-1)d_2d_3\left|\mathrm{E}\left[\frac{n^2}{S_{n-3}^2}\right] - \frac{1}{(\mathrm{E}[D])^2}\right|,
\end{aligned} \tag{18}$$

where $\mathrm{E}[(n^2/S_{n-3}^2)] = \sum_{s=n-3}^{\infty}\frac{n^2}{s^2}f_{S_{n-3}}(s)$. By Lemma 2(ii), it follows that $\lim_{n\to\infty} I_{1n} = 0$ and $\lim_{n\to\infty} I_{2n} = 0$ (i.e, $\lim_{n\to\infty}\sup_{s\in\mathcal{S}_n} s\tilde{p}_2(d_1,d_2,d_3,s) = 0$). Additionally, $\lim_{n\to\infty} I_{3n} = 0$ holds because $\lim_{n\to\infty}\mathrm{E}[(n^2/S_{n-3}^2)] = \lim_{n\to\infty}\mathrm{E}[(n^2/S_{n-2}^2)] = \lim_{n\to\infty}\mathrm{E}[(1/Y_n^2)] = 1/(\mathrm{E}[D])^2$ holds by the LDCT with $1/Y_n^2 \to \mathrm{E}[D]$ w.p.1 as $n\to\infty$ and $1/Y_n^2 \le 9$ is bounded w.p.1 for all $n\ge 3$.

To finish the proof of Lemma 1(i), we need to consider $\lim_{n\to\infty} n^{m-1}P(\mathrm{NB}_{m,n},\mathrm{SLM}_{m,n}^c|d_1,\ldots,d_m)$ for $m=2,3$. For this, we require some additional notation. Before any random wiring in a generic pre-erasure CM, suppose that nodes $A_1, \ldots, A_m$ (fixed $m = 2, 3$) have given values of stubs $a_1, \cdots, a_m$ and that s is the combined number of stubs among all other nodes in the graph, where $a_1, \cdots, a_m, s \geq 0$ are integers with $a_1 + \cdots + a_m + s \geq 2$ even. Let $q_m(a_1, \ldots, a_m, s)$ denote the probability (conditional on $a_1, \ldots, a_m, s$) that, upon random pairing, node A_1 is a neighbor of nodes A_2 through A_m and that no node among $A_1, \ldots, A_m$ has any self-loops or multi-edges and let $\tilde{q}_m(a_1, \ldots, a_m, s)$ denote the probability (conditional on $a_1, \ldots, a_m, s$) that node A_1 is not a neighbor of any of the nodes A_2 through A_m and that no node among $A_1, \ldots, A_m$ has any self-loops or multi-edges. If $m = 2$ with given stub counts $a_1, a_2 \geq 0$ for nodes A_1, A_2 and $s \geq 1$ stubs remaining, then

$$q_2(a_1, a_2, s) = a_1 \frac{a_2}{s + a_1 + a_2 - 1} \tilde{q}_2(a_1 - 1, a_2 - 1, s);$$

that is, if we pick and fix one stub of A_1, the probability that the chosen stub wires with a stub from node A_2 is $a_2/(s+a_1+a_2-1)$ and, conditional on this connection, the probability that the remaining stubs of A_1 and A_2 form no further connections or self-loops or multi-edges is $\tilde{q}_2(a_1 - 1, a_2 - 1, s)$. If $m = 3$ with given stub counts $a_1, a_2, a_3 \geq 0$ for nodes A_1, A_2, A_3 and $s \geq 2$ stubs remaining, then

$$q_3(a_1, a_2, a_3, s) = a_1(a_1 - 1) \frac{a_2}{s + a_1 + a_2 + a_3 - 1} \frac{a_3}{s + a_1 + a_2 + a_3 - 3} \tilde{q}_3(a_1 - 2, a_2 - 1, a_3 - 1, s);$$

that is, given a_1, a_2, s, if we pick and fix two stubs of A_1, the probability that the first chosen stub of A_1 wires with a stub from node A_2 is $a_2/(s + a_1 + a_2 + a_3 - 1)$ and, conditional on this connection, the probability that the second chosen stub of A_1 wires with a stub from node A_3 is $a_3/(s + a_1 + a_2 + a_3 - 3)$ and, conditional on these connections, the probability that the remaining stubs of A_1 form no further edges with A_2 and A_3 with no self-loops or multi-edges among A_1, A_2, A_3 is $\tilde{q}_3(a_1 - 2, a_2 - 1, a_3 - 1, s)$.

Similarly to (14), we may write

$$P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \ldots, d_m) = \sum_{s=n-m}^{\infty} f_{S_{n-m}}(s) P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \ldots, d_m, s), \tag{19}$$

using the conditional probability $P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \ldots, d_m, s)$ given $D_{1,n} = d_1, \ldots, D_{m,n} = d_m, S_{n-m} = s$, which can be written as

$$\begin{aligned} & P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n} | d_1, \ldots, d_m, s) \qquad (20) \\ = & \begin{cases} q_m(d_1, \ldots, d_m, s) & \text{even } d_1 + \cdots + d_m + s \\ \frac{n-m}{n} q_m(d_1, \ldots, d_m, s+1) & \text{odd } d_1 + \cdots + d_m + s \end{cases} \\ = & \begin{cases} \frac{d_1 d_2}{s+d_1+d_2-1} r_2(d_1 - 1, d_2 - 1, s) & m = 2 \\ \frac{d_1(d_1-1)d_2 d_3}{(s+d_1+d_2-1)(s+d_1+d_2-2)} r_3(d_1 - 2, d_2 - 1, d_3 - 1, s) & m = 3 \end{cases} \end{aligned}$$

with

$$r_m(d_1, \ldots, d_m, s) \equiv \begin{cases} \tilde{q}_m(d_1, \ldots, d_m, s) & \text{even } d_1 + \cdots + d_m + s \\ \frac{n-m}{n} \tilde{q}_m(d_1, \ldots, d_m, s+1) & \text{odd } d_1 + \cdots + d_m + s, \end{cases} \qquad m = 2, 3,$$

accounting for the fact that, if $d_1 + \cdots + d_m + s$ is odd, the event $\mathrm{SLM}^c_{m,n}$ requires that given stubs $d_1, \ldots, d_m$ are not incremented by 1, which happens with probability $(n - m)/n$.

We may now show $\lim_{n\to\infty} nP(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n} | d_1, d_2) = d_1 d_2/[\mathrm{E}[D]]$ for $m = 2$ Lemma 1(i). We assume $d_1, d_2 \geq 1$. Using (19) and (20), write

$$\left| nP(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n} | d_1, d_2) - \frac{d_1 d_2}{\mathrm{E}[D]} \right| \leq J_{1n} + J_{2n}$$

where

$$\begin{aligned}
J_{1n} &\equiv \sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s) \left| sP(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n} | d_1, d_2, s) - d_1 d_2 \right| \\
&\leq \frac{n}{n-2} d_1 d_2 \sum_{s=n-2}^{\infty} f_{S_{n-2}}(s) \left[1 - \frac{s}{d_1 d_2} P(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n} | d_1, d_2, s) \right] \\
&\leq \frac{n}{n-2} d_1 d_2 \left(1 - \sum_{s=n-2}^{\infty} f_{S_{n-2}}(s) r_2(d_1 - 1, d_2 - 1, s) \right) \\
J_{2n} &\equiv d_1 d_2 \left| \mathrm{E}\left[\frac{n}{S_{n-2}} \right] - \frac{1}{\mathrm{E}[D]} \right|
\end{aligned}$$

using above $sP(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n}|d_1, d_2, s)/[d_1 d_2] \leq r_2(d_1 - 1, d_2 - 1, s) \leq 1$ for $s \geq n - 2$ with $\sum_{s=n-2}^{\infty} f_{S_{n-2}}(s) = 1$ and recalling $\mathrm{E}[(n/S_{n-2})] = \sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s)$. As in (17), $\lim_{n\to\infty} J_{2n} = 0$. Also, $\mathrm{E}[r_2(d_1 - 1, d_2 - 1, S_{n-2})] \equiv \sum_{s=n-2}^{\infty} f_{S_{n-2}}(s) r_2(d_1 - 1, d_2 - 1, s)$ is the probability that, in a pre-erasure CM with n nodes, two given nodes with initial stub values $d_1 - 1, d_2 - 1$ are not neighbors of each other, have no self-loops or multi-edges, and do not have stubs incremented by 1 (while conditioned on d_1 and d_2 values, this probability is unconditional with respect to the sum of all remaining initial stubs (e.g., S_{n-2}) in the graph). It follows from arguments in Britton, Deijfen and Martin-Löf ([16], Theorem 1) that $\lim_{n\to\infty} \mathrm{E}[r_2(d_1 - 1, d_2 - 1, S_{n-2})] = 1$ since $\mathrm{E}[D] < \infty$, so that $\lim_{n\to\infty} J_{1n} = 0$. Now the limit of $nP(\mathrm{NB}_{2,n}, \mathrm{SLM}^c_{2,n}|d_1, d_2) = d_1 d_2/[\mathrm{E}[D]]$ is established $m = 2$ in Lemma 1(i).
To show $\lim_{n\to\infty} n^2 P(\mathrm{NB}_{3,n}, \mathrm{SLM}^c_{2,n}|d_1, d_2, d_3) = d_1(d_1 - 1)d_2 d_3/[\mathrm{E}[D]]^2$ in Lemma 1(i) for $m = 3$, the argument is analogous

$$\left| n^2 P(\mathrm{NB}_{3,n}, \mathrm{SLM}^c_{3,n}|d_1, d_2, d_3) - \frac{d_1(d-1)d_2 d_3}{[\mathrm{E}[D]]^2} \right| \leq J_{1n} + J_{2n}$$

where

$$\begin{aligned}
J_{1n} &\equiv \sum_{s=n-3}^{\infty} \frac{n^2}{s^2} f_{S_{n-3}}(s) \left| s^2 P(\mathrm{NB}_{3,n}, \mathrm{SLM}^c_{3,n}|d_1, d_2, s) - d_1(d_1 - 1) d_2 d_3 \right| \\
&\leq \left(\frac{n}{n-3} \right)^2 d_1(d_1 - 1) d_2 d_3 \left(1 - \sum_{s=n-3}^{\infty} f_{S_{n-3}}(s) r_3(d_1 - 2, d_2 - 1, d_3 - 1, s) \right) \\
J_{2n} &\equiv d_1(d_1 - 1) d_2 d_3 \left| \mathrm{E}\left[\frac{n^2}{S^2_{n-3}} \right] - \frac{1}{[\mathrm{E}[D]]^2} \right|
\end{aligned}$$

and $\lim_{n\to\infty} J_{2n} = 0$ as in (18) and $\lim_{n\to\infty} J_{1n} = 0$ by $\mathrm{E}[r_3(d_1 - 2, d_2 - 1, d_3 - 1, S_{n-3})] \equiv \sum_{s=n-3}^{\infty} f_{S_{n-3}}(s) r_3(d_1 - 2, d_2 - 1, d_3 - 1, s) \to 1$ as $n \to \infty$.

B.0.2 Proof of Lemma 1(ii)

To establish Lemma 1(ii) for $m = 2$, recall that $p_1(a_1, a_2, s)$ in (12) represents the probability that, under random pairing in the pre-erasure CM, a node A_1 with a_1 stubs shares at least one edge with a second node A_2 with a_2 stubs (where s is the node number of remaining stubs and $a_1 + a_2 + s$ is even). If node A_1's stubs are labeled $j = 1, \ldots, a_1$ and if $p_1^{(j)}(a_1, a_2, s) = a_2/(s + a_1 + a_2 - 1)$ denotes the probability that stub j of node A_1 pairs with some stub from the second node A_2, then we may bound $p_1(a_1, a_2, s) \leq \sum_{j=1}^{a_1} p_1^{(j)}(a_1, a_2, s) = a_1 a_2/(s + a_1 + a_2 - 1)$ for $a_1, a_2 \geq 0$. Using this with (14) and (15), it then follows that for $d_1, d_2 \geq 1$ and $n \geq 3$

$$nP(\mathrm{NB}_{2,n}|d_1, d_2) \leq 2d_1 d_2 \mathrm{E}\left(\frac{n}{S_{n-2}}\right) \leq C d_1 d_2$$

with $C = 2\sup_{n\geq 3} \mathrm{E}[(n/S_{n-2})] = O(1)$ since $\mathrm{E}[n/S_{n-2}] = \sum_{s=n-2}^{\infty} \frac{n}{s} f_{S_{n-2}}(s) \to 1/\mathrm{E}[D] < \infty$ (by the SLLN under $\mathrm{E}[D] = O(1)$ as in (20)).

B.0.3 Proof of Lemma 1(iii)

To show Lemma 1(iii) for $m = 3$, consider first bounding $n^2 P(\mathrm{NB}_{3,n}|d_1, d_2, d_3)$, $d_2, d_3 \geq d_1$, with an argument analogous to the previous one. That is, recalling $p_2(a_1, a_2, a_3, s)$ from (13), if node A_1's stubs are labeled $j = 1, \ldots, a_1$ and if $p_2^{(i,j)}(a_1, a_2, a_3, s) = [a_2/(s+a_1+a_2+a_3-1)][a_3/(s+a_1+a_2+a_3-3)]$ denotes the probability that stub i of node A_1 wires with some stub from the second node A_2 (having a_2 stubs) and stub j wires with some stub from the third node A_3 (having a_3) stubs, then we may bound $p_2(a_1, a_2, a_3, s) \leq \sum_{i=1}^{a_2} \sum_{j=1, j\neq i}^{a_2} p_2^{(i,j)}(a_1, a_2, a_3, s) \leq a_1(a_1 - 1)a_2 a_3/[(s + a_1 + a_2 + a_3 - 1)(s + a_1 + a_2 + a_3 - 3)]$. From this and (14) with (16), it follows that for $d_2, d_3 \geq d_1 \geq 1$ and $n \geq 4$

$$n^2 P(\mathrm{NB}_{3,n}|d_1, d_2, d_3) \leq 2d_1^2 d_2 d_3 \mathrm{E}\left[\frac{n^2}{S_{n-3}}\right] \leq C(d_1 d_2 d_3)^{4/3} \tag{21}$$

with $C = 2\sup_{n\geq 3} \mathrm{E}[n^2/S_{n-3}^2] = O(1)$ since $\mathrm{E}[n^2/S_{n-3}^2] = \sum_{s=n-3}^{\infty} \frac{n^2}{s^2} f_{S_{n-3}}(s) \to 1/[\mathrm{E}[D]]^2 < \infty$ as in (17).

To complete the proof of Lemma 1(iii) for $m = 3$, we show

$$n^2 P(\mathrm{NBD}_{3,n}, \mathrm{SLM}_{3,n}|d_1, d_2, d_3) \prod_{i=1}^{3} f_n(d_i) \leq C(d_1 d_2 d_3)^{4/3} \prod_{i=1}^{3} f(d_i) \tag{22}$$

holds for some $C > 0$, not depending on $d_1, d_2, d_3 \geq 1$ or $n \geq 4$. The constant C will be determined as $C = \max\{H_1, H_2, H_3, H_4\}$, where $H_1, \ldots, H_4$ are constants specified by case analysis below. Note that $n^2 P(\mathrm{NBD}_{3,n}, \mathrm{SLM}_{3,n}|d_1, d_2, d_3) \leq n^2 P(\mathrm{NB}_{3,n}|d_1, d_2, d_3)$ for all $d_1, d_2, d_3 \geq 1$, so that for all $n \geq 4$

$$n^2 P(\mathrm{NBD}_{3,n}, \mathrm{SLM}_{3,n}|d_1, d_2, d_3) \prod_{i=1}^{3} f_n(d_i) \leq H_1(d_1 d_2 d_3)^{4/3} \prod_{i=1}^{3} f(d_i)$$

follows for $d_2, d_3 \geq d_1$ by (21) and by $f_n(d) \leq f(d) \sup_{n\geq 1} C_n^{-1}$, $d \geq 1$ from (3), so that H_1 does not depend on $n \geq 4$ or $d_2, d_3 \geq d_1$. Consider d_1, d_2, d_3 where $\min\{d_1, d_2, d_3\} < d_1$ and, for $\tau \in (0, 1/2)$ from (3), pick and fix a large integer $M > 1$ where $\tau 2M + 2 < M$. If $d_1 - \min\{d_1, d_2, d_3\} \leq 5M$, then (21) yields

$$n^2 P(\mathrm{NBD}_{3,n}, \mathrm{SLM}_{3,n}|d_1, d_2, d_3) \leq n^2 P(\mathrm{NB}_{3,n}|d_1, d_2, d_3) \leq H_2(d_1 d_2 d_3)^{4/3}$$

for $H_2 = (10M)^{2/3} 2\sup_{n\geq 3} \mathrm{E}[n^2/S_{n-3}^2] = O(1)$ not depending on $n \geq 4$ or $d_1, d_2, d_3 \geq 1$ with $d_1 - \min\{d_1, d_2, d_3\} \leq 5M$. Finally, consider the possibility $d_1 - \min\{d_1, d_2, d_3\} > 5M$. In this case, given $D_{1,n} = d_1, D_{2,n} = d_2, D_{3,n} = d_3$, in order that event $\mathrm{NBD}_{3,n}$ in the ECM graph hold (i.e., $D_{2,n}^s \geq D_{1,n}^s$ and $D_{3,n}^s \geq D_{1,n}^s$ hold in addition to node 1 being a neighbor of nodes 2 and 3), node 1 needs at least $4M$ of its $d_1 > 5M + 1$ stubs to be erased (even if the stubs from one node among nodes $1, 2$ or 3 is incremented by 1 when $\sum_{i=1}^{n} D_{i,n}$ is odd), implying either event $\mathrm{NB}_{3,n}$ holds with node 1 having at least M self-loops in the pre-erasure CM or event $\mathrm{NB}_{3,n}$ holds with node 1 having at least M multi-edges in the pre-erasure CM. Hence, if $d_1 - \min\{d_1, d_2, d_3\} > 5M$,

$$
\begin{aligned}
P(\mathrm{NBD}_{3,n}, \mathrm{SLM}_{3,n}|d_1, d_2, d_3) \quad \leq \quad & P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ self-loops}|d_1, d_2, d_3) \\
& + P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ multi-edges}|d_1, d_2, d_3),
\end{aligned}
$$

Hence, the proof of (22) will be complete upon establishing that

$$
\begin{aligned}
& n^2 P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ self-loops}|d_1, d_2, d_3) \prod_{i=1}^{3} f_n(d_i) && (23) \\
\leq \quad & H_3 (d_1 d_2 d_3)^{4/3} \prod_{i=1}^{3} f(d_i) \\
& n^2 P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ multi-edges}|d_1, d_2, d_3) \prod_{i=1}^{3} f_n(d_i) && (24) \\
\leq \quad & H_4 (d_1 d_2 d_3)^{4/3} \prod_{i=1}^{3} f(d_i)
\end{aligned}
$$

for H_3, H_4 not depending on $n \geq 4$ or $d_1, d_2, d_3 \geq 1$ with $d_1 - \min\{d_1, d_2, d_3\} > 5M$.

We consider establishing (23) and, supposing given values $D_{1,n} = d_1, D_{2,n} = d_2, D_{3,n} = d_3, S_{n-3} = s$, we first bound the corresponding conditional probability of the event "$\mathrm{NB}_{3,n}$, node 1 has at least M self-loops" when $n \geq 4$ and $d_1 - \min\{d_1, d_2, d_3\} > 5M$. Accounting for the possibility that a node among $\{1, 2, 3\}$ might have its stubs incremented by 1, there are at most $(d_1+1)d_1$ ways to pick and fix two stubs of node 1 to wire to one of the (at most) $d_2 + 1$ stubs of node 2 and one of the (at most) $d_3 + 1$ stubs of node 3, leaving at most $P_{2k}^{d_1-1}/[k!2^k]$ ways to pick and fix $2k$ pairs of stubs from node 1 to form k self-loops, $k = 0, \ldots, \lfloor (d_1 - 1)/2 \rfloor$. After designating two stubs of node 1 to wire to a stub of node 2 and a stub of node 3 and designating $2k$ (fixed k) stubs of node 1 to form self-loops, the probability that this wiring event actually occurs in the pre-erasure CM is at most $[(d_2+1)/s][(d_3+1)/s]s^{-k}$ (derived from conditioning probabilities). Hence,

$$
\begin{aligned}
& n^2 P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ self-loops}|d_1, d_2, d_3, s) \\
\leq \quad & (d_1+1)d_1 \frac{(d_2+1)}{s} \frac{(d_3+1)}{s} \sum_{k=M}^{\lfloor (d_1-1)/2 \rfloor} \frac{1}{k! 2^k} P_{2k}^{d_1-1} \frac{1}{s^k}
\end{aligned}
$$

so that, noting $P_{2k}^{d_1-1} \leq d_1^{2k}$, it follows that

$$
\begin{aligned}
& n^2 P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ self-loops}|d_1, d_2, d_3) \prod_{i=1}^{3} f_n(d_i) \\
= \quad & n^2 \sum_{s=n-3}^{\infty} P(\mathrm{NB}_{3,n}, \text{node 1 has at least } M \text{ self-loops}|d_1, d_2, d_3, s) f_{S_n}(s) \prod_{i=1}^{3} f_n(d_i) \\
\leq \quad & n^2 \left(\prod_{i=1}^{3} f_n(d_i) \right) \sum_{s=n-3}^{\infty} (d_1+1)d_1 \frac{(d_2+1)}{s} \frac{(d_3+1)}{s} f_{S_n}(s) \sum_{k=M}^{\lfloor (d_1-1)/2 \rfloor} \frac{1}{k! 2^k} P_{2k}^{d_1-1} \frac{1}{s^k} \\
\leq \quad & 8 d_1^{4/3} d_2 d_3 \left(\prod_{i=1}^{3} f_n(d_i) \right) \sum_{s=n-3}^{\infty} f_{S_n}(s) \frac{n^2 d_1^{2M+2/3}}{s^{M+2}} \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{d_1^{2k}}{s^k}.
\end{aligned}
$$

From this, (23) holds with $H_3 = \max\{N_1^2, 162^{2+M} \sup_{n\geq 1} C_n^{-3}\}$ using that $f_n(d) = 0$ for $d > L(n)n^\tau$ and that there exists an integer $N_1 \geq 6$ such that $L(n)n^\tau \leq n^{1/2}$ and $[L(n)n^\tau]^{2M+2/3} \leq n^M$ hold for $n \geq N_1$ since $L(\cdot)$ is slowly varying and $\tau \in (0, 1/2)$, $\tau 2M + 2 < M$.

We next consider establishing (24) for $n \geq 4$ and $d_1 - \min\{d_1, d_2, d_3\} > 5M$. We require some additional notation. Suppose $1 < v_1 < \ldots < v_w \subset \{2, \ldots, n\}$ denote the indices of $2 \leq w \leq M+2$ nodes in a size n (pre-erasure) CM graph where we set $v_1 = 2, v_2 = 3$ and let k_i, $i = 1, \ldots, w$ denote the number of edges between node 1 and node v_i. In order for nodes $v_1 = 2$ and $v_2 = 3$ to be neighbors of node 1 and for node 1 to have at least M multi-edges in a graph with n nodes under the pre-erasure CM, there must exist some subset $\{v_1, \ldots, v_w\} \subset \{2, \ldots, n\}$ of w nodes with $2 \leq w \leq M+2$ for which it holds that $k_1 \geq 1, k_2 \geq 1$ (to ensure nodes 2 and 3 are neighbors of 1), $k_i \geq 2$ for $i = 3, \ldots, w$ when $w > 2$ (to ensure nodes 1 and v_i have at least one multi-edge) and $-2 + \sum_{i=1}^w k_i \geq M$ (to ensure at least M multi-edges for node 1 based on the w nodes $v_1, \ldots, v_w$). Or expressed in terms of set notation, we have

$$\text{"NB}_{3,n}\text{, node 1 has at least } M \text{ multi-edges"} \subset \bigcup_{w=2}^{M+2} \bigcup_{(v_1,\ldots,v_w)\in\mathcal{B}_w} \bigcup_{(k_1,\ldots,k_w)\in\mathcal{C}_w} \bigcap_{i=1}^{w} E_{v_i,k_i,n}. \tag{25}$$

where above $E_{v,k,n}$ denotes the event that node $1 < v \leq n$ has exactly $k \geq 1$ edges after wiring in a pre-erasure CM with n nodes and we define index sets $\mathcal{B}_2 = \{(2,3)\}$, $\mathcal{C}_2 = \{(k_1, k_2) :$ integer $k_1, k_2 \geq 1;\ k_1 + k_2 - 2 \geq M\ \}$ and $\mathcal{B}_w = \{(v_1, \ldots, v_w) : v_1 = 2, v_2 = 3 < v_3 < \cdots < v_w \leq n\}$, $\mathcal{C}_w = \{(k_1, \ldots, k_w) :$ integer $k_1, k_2 \geq 1; k_3, \ldots, k_w \geq 2; \sum_{i=1}^w k_i - 2 \geq M\ \}$ for $w \geq 2$. To establish (24) for $n \geq 4$ and $d_1 - \min\{d_1, d_2, d_3\} > 5M$, we shall use (25) and the fact that there exists an integer $N_2 \geq \max\{8, M+3\}$ such that $1 \leq 2[n^\tau L(n)] \leq (n-4)^{1/2}$, $[n^\tau L(n)]^{2M} \leq (n-4)^{M-2}$ (since $\tau 2M + 2 < M$ and $L(\cdot)$ is slowly varying) and $(n-3) - M[L(n)n^\tau] \geq (n-4)$ for $n \geq N_2$. (For a size n CM graph (assuming $f(0) = 0$), $(n-3)$ is the smallest value of $\sum_{i=3}^n D_{i,n}$ while $M[L(n)n^\tau]$ is the largest possible value of $\sum_{i=4}^n D_{i,n}$ under $f_n(\cdot)$; that is, $\sum_{i=M+3}^n D_{i,n}$ is large and at least $(n-4)$ for $n \geq N_2$.)

Now fix $2 \leq w \leq M+2$, fix $(v_1, \ldots, v_w) \in \mathcal{B}_w$ and fix $(k_1, \ldots, k_w) \in \mathcal{C}_w$ (with $v_1 = 2, v_2 = 3$). Then, it holds for any $w > 2$ that

$$\begin{aligned} & P\left(\bigcap_{i=1}^{w} E_{v_i,k_i,n} | D_1 = d_1, D_2 = d_3, D_3 = d_3\right) \prod_{i=1}^{3} f_n(d_i) \qquad (26) \\ = & \sum_{d_{v_3}=k_3}^{L(n)n^\tau} \cdots \sum_{d_{v_w}=k_w}^{L(n)n^\tau} \left(f_n(d_1) \prod_{j=1}^{w} f_n(d_{v_i})\right) P\left(\bigcap_{i=1}^{w} E_{v_i,k_i,n} | D_1 = d_1, D_{v_1} = d_{v_1}, \ldots, D_{v_w} = d_{v_w}\right). \end{aligned}$$

Then, for $n \geq N_2$ and $w \geq 2$, it follows (using conditioning probabilities with $\sum_{i=M+3}^n D_{i,n} \geq (n-3) - M[L(n)n^\tau] \geq (n-4)$ and allowing for the possibility that a stub count among $d_1, d_{v_1}, \ldots, d_{v_w}$

may be incremented by 1) that

$$
\begin{aligned}
& n^w P\left(\bigcap_{i=1}^{w} E_{v_i,k_i,n} | D_1 = d_1, D_{v_1} = d_{v_1}, \ldots, D_{v_w} = d_{v_w}\right)\left(f_n(d_1)\prod_{j=1}^{w} f_n(d_{v_i})\right) \qquad (27)\\
\leq\ & n^w f_n(d_1)\prod_{j=1}^{w} 4 f_n(d_{v_j})\left(\frac{d_1 d_{v_j}}{n-4}\right)^{k_j}\\
\leq\ & \left(\frac{4n}{(n-4)C_n}\right)^w \left(f(d_1)\prod_{j=1}^{w} f(d_{v_j})\right) d_1^{4/3}\left(\prod_{j=1}^{w} d_{v_j}\right)^{4/3} \frac{[n^\tau L(n)]^{-4/3(w-1)+2\sum_{j=1}^{w} k_j - k_1^* - k_2^*}}{(n-4)^{-w+\sum_{j=1}^{w} k_i}}\\
\leq\ & \tilde{H}_4\left(f(d_1)\prod_{j=1}^{w} f(d_{v_j})\right) d_1^{4/3}\left(\prod_{j=1}^{w} d_{v_j}\right)^{4/3}\left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M}, \quad \tilde{H}_4 \equiv (8/C_n)^{M+2},
\end{aligned}
$$

where above we use $k_i^* = \min\{k_i, 4/3\}$ for $i = 1, 2$; $n/(n-4) \leq 2$ for $n \geq N_2$; and

$$
\begin{aligned}
\frac{[n^\tau L(n)]^{-4/3(w-1)+2\sum_{j=1}^{w} k_j - k_1^* - k_2^*}}{(n-4)^{-w+\sum_{j=1}^{w} k_i}} &\leq \begin{cases} \left(\frac{n^\tau L(n)}{(n-4)^{1/2}}\right)^{2\sum_{j=1}^{w} k_j - 2w} & w \geq 3 \\ \frac{[n^\tau L(n)]^{2M}}{(n-4)^{M-2}}\left(\frac{n^\tau L(n)}{(n-4)^{1/2}}\right)^{2\sum_{j=1}^{w} k_j - 2M} & w = 2 \end{cases}\\
&\leq \left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M}
\end{aligned}
$$

by $1 \leq 2[n^\tau L(n)] \leq (n-4)^{1/2}$ and $[n^\tau L(n)]^{2M} \leq (n-4)^{M-2}$ for $n \geq N_2$ along with $-4(m-1)/3 - k_1 * -k_2^* \leq -2w$ for $w \geq 3$. Hence, by (26)-(27), we have

$$
\begin{aligned}
& n^w P\left(\bigcap_{i=1}^{w} E_{v_i,k_i,n} | D_1 = d_1, D_2 = d_3, D_3 = d_3\right)\prod_{i=1}^{3} f_n(d_i)\\
\leq\ & \begin{cases} \tilde{H}_4\left(\prod_{i=1}^{3} d_i^{4/3} f(d_i)\right)\left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M} & w = 2 \\ \sum_{d_{v_3}=0}^{\infty}\cdots\sum_{d_{v_w}=0}^{\infty} \tilde{H}_4\left(f(d_1)\prod_{j=1}^{w} f(d_{v_j})\right) d_1^{4/3}\left(\prod_{j=1}^{w} d_{v_j}\right)^{4/3}\left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M} & w = 3 \end{cases}\\
=\ & \tilde{H}_4[\mathrm{E}[D^{4/3}]]^{w-2}(d_1 d_2 d_3)^{4/3}\prod_{i=1}^{3} f(d_i)\left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M} \qquad (28)
\end{aligned}
$$

for $n \geq N_2$ and any fixed $2 \leq w \leq M+2$, $(v_1, \ldots, v_w) \in \mathcal{B}_w$ $(k_1, \ldots, k_w) \in \mathcal{C}_w$ (with $v_1 = 2, v_2 = 3$). Then, by (25) and (28), we have for $d_1, d_2, d_3 \geq 1$ with $d_1 - \min\{d_1, d_2, d_3\} > 5M$ and $n \geq N_2$,

$$
\begin{aligned}
& n^2 P(\mathrm{NB}_{3,n}, \text{node 1 has 1 has at least } M \text{ multi-edges} | d_1, d_2, d_3)\prod_{i=1}^{3} f_n(d_i)\\
\leq\ & \sum_{w=2}^{M+2}\frac{1}{n^{w-2}}\sum_{(v_1,\ldots,v_w)\in\mathcal{B}_w}\ \sum_{(k_1,\ldots,k_w)\in\mathcal{C}_w} n^w P(\bigcap_{i=1}^{w} E_{v_i,k_i,n} | D_1 = d_1, D_2 = d_3, D_3 = d_3)\\
\leq\ & \tilde{H}_4(d_1 d_2 d_3)^{4/3}\prod_{i=1}^{3} f(d_i)\sum_{w=2}^{M+2}[\mathrm{E}[D]]^{w-2}\frac{|B_w|}{n^{w-2}}\sum_{k_1=0}^{\infty}\cdots\sum_{k_w=0}^{\infty}\left(\frac{1}{2}\right)^{\sum_{j=1}^{w} k_j - 2M}\\
\leq\ & \tilde{H}_4 2^{3M+2}[\mathrm{E}[D]]^M (d_1 d_2 d_3)^{4/3}\prod_{i=1}^{3} f(d_i),
\end{aligned}
$$

using $|\mathcal{B}_w| \le n^{w-2}$ to bound the cardinality of $\mathcal{B}_w$. Now (24) follows with $H_4 = \max\{N_2^2, \tilde{H}_4 2^{3M+2}[\mathrm{E}[D]]^M\}$ not depending on $n \ge 4$ or $d_1, d_2, d_3 \ge 1$ with $d_1 - \min\{d_1, d_2, d_3\} > 5M$. The proof of Lemma 1 is finished, under the assumption that $f(0) = 0$ in (1).

B.0.4 Modified proof of Lemma 1

The previous proof of Lemma 1 assumed $f(0) = 0$, and here we provide a modification that shows that Lemma 1 still holds in the case that $0 < f(0) < 1$ (since $\mathrm{E}[D] > 0$ by assumption, $f(0) < 1$ must hold). To see this, recall $S_{n-m} = \sum_{i=m+1}^{n} D_{i,n}$ for $m = 2, 3$ and define $a_{n-m} \equiv 2^{-1}\mathrm{E}[D_{1,n}] = 2^{-1}\sum_{d=0}^{n^\tau L(n)} d \cdot f_n(d)$ for $n > m$. Note that $\mathrm{E}[S_{n-m}] = 2a_{n-m}(n-m)$ and there exists some $\epsilon > 0$ such that $a_{n-m} > \epsilon$ for $n > m$. Now pick a large integer K such that $\tau 2K + 2 < K$ (possible since $\tau \in (0, 1/2)$). By Markov's inequality and the fact that $D_{1,n}, \ldots, D_{n,n}$ are independent and identically distributed and bounded by $n^\tau L(n)$, it holds for $m = 2$ or 3 that

$$\begin{aligned}
n^{m-1}P(S_{n-m} \le a_{n-m}(n-m)) &\le n^{m-1}P\left(|S_{n-m} - \mathrm{E}[S_{n-m}]|^{2K} \ge [a_{n-m}(n-m)]^{2K}\right) \\
&\le \frac{n^2}{[a_{n-m}(n-m)]^{2K}}\mathrm{E}\left[|S_{n-m} - \mathrm{E}S_{n-m}|^{2K}\right] \\
&\le \frac{Cn^2}{\epsilon^{2K}(n-m)^{2K}}(n-m)^K \mathrm{E}[D_{1,n}]^{2K} \\
&\le \frac{C}{\epsilon^{2K}} n^{2K\tau+2-K}[L(n)]^{2K}
\end{aligned}$$

for a constant $C > 0$ not depending on $n \ge 4$. Note that $\lim_{n\to\infty} n^{2K\tau+2-K}[L(n)]^{2K} = 0$ (since $2K\tau + 2 - K < 0$ and $L(\cdot)$ is slowly varying) so that

$$\lim_{n\to\infty} n^{m-1}P(S_{n-m} \le a_{n-m}(n-m)) = 0 \tag{29}$$

for $m = 2, 3$.

In the proof of Lemma 1 from Section B which assumed $f(0) = 0$, the sum $S_{n-m}/(n-m) \ge 1$ (for $m = 2, 3$) was bounded away from zero (implying that S_{n-m} diverged to ∞ necessarily as $n \to \infty$), which was a key component in that proof. Now the imposed additional event $S_{n-m}/(n-m) \ge a_{n-m} \ge \epsilon$ in (30) forces the sum S_{n-m} to grow as $n \to \infty$, so at the very least S_{n-m} must be greater than $n\epsilon$. For $m = 2$ or 3, all of the bounds and limits for conditional probabilities $n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}_{m,n}|d_1, \ldots, d_m)$, $n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}|d_1, \ldots, d_m)$ and $n^{m-1}P(\mathrm{NB}_{m,n}|d_1, \ldots, d_m)$ stated in Lemma 1 can be established for analogous quantities

$$\begin{gathered}
n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}_{m,n}, S_{n-m} \ge a_{n-m}(n-m)|d_1, \ldots, d_m), \\
n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}, S_{n-m} \ge a_{n-m}(n-m)|d_1, \ldots, d_m), \\
n^{m-1}P(\mathrm{NB}_{m,n}, S_{n-m} \ge a_{n-m}(n-m)|d_1, \ldots, d_m).
\end{gathered} \tag{30}$$

To show Lemma 1 holds for quantities in (30), the only serious modifications to the previous proof of Lemma 1 occur in steps (17) and (18), where now we require

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{n^{m-1}}{S_{n-m}^{m-1}}\mathbb{I}\left[S_{n-m} \ge a_{n-m}(n-m)\right]\right] \equiv \lim_{n\to\infty} \sum_{s \ge a_{n-m}(n-m)} \frac{n^{m-1}}{s^{m-1}} f_{S_{n-m}}(s) = \frac{1}{[\mathrm{E}[D]]^{m-1}}$$

for $m = 2, 3$ (using an indicator function $\mathbb{I}(\cdot)$ above) rather than $\lim_{n\to\infty} \mathrm{E}[n^{m-1}/S_{n-m}^{m-1}] = \lim_{n\to\infty} \sum_{s=n-m}^{\infty} \frac{n^{m-1}}{s^{m-1}} f_{S_{n-m}}(s) =$ $1/[\mathrm{E}[D]]^{m-1}$ in (17) and (18). To establish these alternative limits for $m = 2, 3$, note that the proof of (17) showed that there exist random variables Y_n having the same distribution as S_{n-2}/n for which $Y_n \to \mathrm{E}[D]$ w.p.1 as $n \to \infty$. Since $a_n \leq (1-\epsilon)\mathrm{E}[D]$, it holds that $\lim_{n\to\infty} Y_n = \mathrm{E}[D] \geq (1-\epsilon)\mathrm{E}[D] \geq \limsup_{n\to\infty} a_n$ w.p.1 implying $\mathbb{I}[Y \leq a_{n-2}(n-2)/n] \to 1$ w.p.1 as $n \to \infty$ as well as $Z_n \equiv 1/Y_n \mathbb{I}[Y \geq a_{n-2}(n-2)/n] \to 1/\mathrm{E}[D]$. Since $Z_n \leq n/[(n-2)a_{n-2}] \leq 2/\epsilon$ w.p.1 for all $n \geq 1$, the LDCT gives for $m = 2$

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{n}{S_{n-2}} \mathbb{I}\left[S_{n-2} \geq a_{n-2}(n-2)\right]\right] = \lim_{n\to\infty} \mathrm{E}[Z_n] = \frac{1}{\mathrm{E}[D]}.$$

Similarly, for $m = 3$,

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{n^2}{S_{n-3}} \mathbb{I}\left[S_{n-3} \geq a_{n-3}(n-3)\right]\right] = \lim_{n\to\infty} \mathrm{E}[Z_n^2] = \frac{1}{[\mathrm{E}[D]]^2}.$$

Finally, if Lemma 1 holds for the quantities in (30), then Lemma 1 will also hold for the conditional probabilities $n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}_{m,n}|d_1,\ldots,d_m)$, $n^{m-1}P(\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}|d_1,\ldots,d_m)$ and $n^{m-1}P(\mathrm{NB}_{m,n}|d_1,\ldots,d_m)$ in the case $0 < f(0) < 1$ under (29) because

$$\begin{aligned}
& n^{m-1}P(\mathcal{A}_{m,n}, S_{n-m} \geq a_{n-m}(n-m)|d_1,\ldots,d_m) \\
\leq\ & n^{m-1}P(\mathcal{A}_{m,n}|d_1,\ldots,d_m) \\
\leq\ & n^{m-1}P(\mathcal{A}_{m,n}, S_{n-m} \geq a_{n-m}(n-m)|d_1,\ldots,d_m) + n^{m-1}P(S_{n-m} \leq a_{n-m}(n-m)|d_1,\ldots,d_m)
\end{aligned}$$

where the generic event $\mathcal{A}_{m,n}$ above could be taken as "$\mathrm{NB}_{m,n}, \mathrm{SLM}_{m,n}$," "$\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}$," or "$\mathrm{NB}_{m,n}$" with $m = 2, 3$. By (29), $n^{m-1}P(\mathcal{A}_{m,n}|d_1,\ldots,d_m)$ will have the same limits in Lemma 1 which hold for $n^{m-1}P(\mathcal{A}_{m,n}, S_{n-m} \geq a_{n-m}(n-m)|d_1,\ldots,d_m)$ for $\mathcal{A}_{m,n}$ taken as "$\mathrm{NB}_{m,n}, \mathrm{SLM}^c_{m,n}$" or "$\mathrm{NB}_{m,n}$" with $m = 2, 3$. Additionally, by (29), $n^{m-1}P(\mathcal{A}_{m,n}|d_1,\ldots,d_m)$ can be bounded by the same quantities given in Lemma 1. (Consider, for illustration, $m = 3$ and $\mathcal{A}_{3,n} = \mathrm{NB}_{3,n}$ and suppose the bound $n^2P(\mathrm{NB}_{3,n}, S_{n-3} \geq a_{n-3}(n-3)|d_1,d_2,d_3) \leq C(d_1d_2d_3)^{4/3}$ of Lemma 1 applies for some $C > 0$ independent of $n \geq 4$ and $d_2, d_3 \geq d_1 \geq 0$. By (29), there exists some $N \geq 4$ such that $n^2P(S_{n-3} \geq a_{n-3}(n-3)) \leq C$ so that $n^2P(\mathrm{NB}_{3,n}|d_1,d_2,d_3) \leq \max\{2C, N^2\}(d_1d_2d_3)^{4/3}$ holds for all $n \geq 4$ and $d_2, d_3 \geq d_1 \geq 0$; note $n^2P(\mathrm{NB}_{3,n}|d_1,d_2,d_3) = 0$ trivially if $d_1 = 0$.) $\square$

B.0.5 Proof of Lemma 2.

Fix values a_1, a_2, a_3. To evaluate the limits in Lemma 2, we use Stirling's approximation for $\log(s!)$ as integer $s \to \infty$ given by

$$\log(s!) = \frac{\log(\sqrt{2\pi})}{2} + s\log s - s + \frac{1}{12s} + O\Big(\frac{1}{s^3}\Big).$$

With this and assuming $a_1 + a_2 + s$ is even, we may expand the permutation functions defining $h(a_1, a_2, s, k)$ in $p_1(a_1, a_2, s)$ from (12) to find

$$\begin{aligned}
h(a_1,a_2,s,k) \quad = \quad & \frac{a_1!\, e^k}{(a_1-2k)!k!2^k} \frac{1}{s^k}\Big(1 - \frac{(a_1-2k)}{s}\Big)^{-(s-(a_1-2k)-1/2}\Big(1 + \frac{a_1+a_2}{s}\Big)^{-(s+a_1+a_2)/2} \\
& \times\Big(1 + \frac{a_2-(a_1-2k)}{s}\Big)^{(s+a_2-(a_1-2k))/2} \exp\left(\frac{4k-3a_1}{12s^2} + a_1a_2\, O\left(s^{-3}\right)\right),
\end{aligned}$$

where $0 \le k \le a_1/2$ is an integer. From this, we obtain the following expansions of $h(a_1, a_2, s, k)$ as $s \to \infty$ (again assuming even $a_1 + a_2 + s$) as a function of $k \le a_1/2$:

$$\begin{aligned}
h(a_1, a_2, s, k) &= O\Big(\frac{1}{s^k}\Big) \quad k \ge 0; \\
h(a_1, a_2, s, 2) &= \frac{a_1(a_1-1)(a_1-2)(a_1-3)}{8s^2} + O\Big(\frac{1}{s^3}\Big); \\
h(a_1, a_2, s, 1) &= \frac{a_1(a_1-1)}{2s} \exp\left[\frac{(a_2+1)(1-a_1)}{s} - \frac{1}{2}\frac{(a_1-2)(a_1-3)}{s} + O\Big(\frac{1}{s^2}\Big)\right]; \\
h(a_1, a_2, s, 0) &= \exp\left[-\frac{a_1(a_1-1)}{2s} - \frac{a_1 a_2}{s} + \frac{1}{s^2}\Big[\frac{a_2^2 a_1}{2} + \frac{a_1(a_1-1)}{4}\Big] + O\Big(\frac{1}{s^3}\Big)\right].
\end{aligned}$$

Now recalling $s^* = s$ if $a_1 + a_2 + s$ is even and $s^* = s + 1$ if $a_1 + a_2 + s$ is odd, the above expressions show

$$\begin{aligned}
\lim_{s\to\infty} s p_1(a_1, a_2, s^*) &= \lim_{s\to\infty} s^* p_1(a_1, a_2, s^*) \\
&= \lim_{s\to\infty} s^* \left(1 - \left[h(a_1, a_2, s^*, 0) + h(a_1, a_2, s^*, 1) + O\Big(\frac{1}{s^2}\Big)\right]\right) \\
&= \lim_{s\to\infty} \left(a_1 a_2 + O\Big(\frac{1}{s}\Big)\right) = a_1 a_2,
\end{aligned}$$

establishing Lemma 2(i).

To show Lemma 2(ii), we similarly have

$$\begin{aligned}
p_1(a_1, a_2, s^*) &= 1 - \left[h(a_1, a_2, s^*, 0) + h(a_1, a_2, s^*, 1) + h(a_1, a_2, s^*, 2) + O\Big(\frac{1}{s^3}\Big)\right] \\
&= \frac{a_1 a_2}{s^*} - \frac{1}{(s^*)^2}\left[\frac{a_2^2(a_1^2 + a_1)}{2} + \frac{a_2 a_1(a_1-1)}{2}\right] + O\Big(\frac{1}{s^3}\Big),
\end{aligned}$$

which yields the following decomposition as $s \to \infty$:

$$\begin{aligned}
p_2(a_1, a_2, a_3, s^*) &= \Big(p_1(a_1, a_2, s^* + a_3) - \frac{a_1 a_2}{s^*}\Big) + \Big(p_1(a_1, a_3, s^* + a_2) - \frac{a_1 a_3}{s}\Big) \\
&\quad - p_1(a_1, a_2 + a_3, s^*) + \frac{a_1(a_2 + a_3)}{s^*} \\
&= \frac{a_1(a_1-1)a_2 a_3}{(s^*)^2} + O\Big(\frac{1}{s^3}\Big).
\end{aligned}$$

Now

$$\lim_{s\to\infty} s^2 p_2(a_1, a_2, a_3, s^*) = \lim_{s\to\infty} (s^*)^2 p_2(a_1, a_2, a_3, s^*) = a_1(a_1-1)a_2 a_3$$

follows, establishing Lemma 2(ii). □